

Insensitizing exact controls for the scalar wave equation and exact controllability of 2-coupled cascade systems of PDE's by a single control.

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Abstract

We study the exact controllability, by a reduced number of controls, of coupled cascade systems of PDE's and the existence of exact insensitizing controls for the scalar wave equation. We give a necessary and sufficient condition for the observability of abstract coupled cascade hyperbolic systems by a single observation, the observation operator being either bounded or unbounded. Our proof extends the two-level energy method introduced in [2, 4] for symmetric coupled systems, to cascade systems which are examples of non symmetric coupled systems. In particular, we prove the observability of two coupled wave equations in cascade if the observation and coupling regions both satisfy the Geometric Control Condition (GCC) of Bardos Lebeau and Rauch [11]. By duality, this solves the exact controllability, by a single control, of 2-coupled abstract cascade hyperbolic systems. Using transmutation, we give null-controllability results for the multidimensional heat and Schrödinger 2-coupled cascade systems under (GCC) and for any positive time. By our method, we can treat cases where the control and coupling coefficients have disjoint supports, partially solving an open question raised by de Teresa [47]. Moreover we answer the question of the existence of exact insensitizing locally distributed as well as boundary controls of scalar multidimensional wave equations, raised by J.-L. Lions [35] and later on by Dáger [19] and Tebou [44].

1 Introduction

The questions of the existence of insensitizing controls for scalar heat and wave equations are challenging issues. J.-L. Lions [35] introduced this notion in 1989 to define controls which are robust to small unknown perturbations

on the initial data. It is by now well-known [35, 19, 12, 47, 13, 14, 48, 44, 45] that the existence of such controls is equivalent to an exact controllability result for an associated 2-coupled cascade of heat (resp. wave, ...) system, for which only one equation is controlled. Similar questions are considered for the Stokes equations in [25] and for the Navier-Stokes equations in [26, 27]. Hence the design of such insensitizing controls is related to the controllability, by a reduced number of controls, of coupled systems. Furthermore, many applicative issues in mechanics, biology or medicine lead also to similar controllability issues for coupled systems, which may be of parabolic, hyperbolic or mixed type. The coupling may also be more general than the cascade form. An increasing number of papers deals with these questions since then.

Coupled parabolic or diffusive control systems of order 2 have the general form

$$\begin{cases} e^{i\theta} y_t - \Delta y + \mathcal{C}y = Bv, \text{ in } Q_T = \Omega \times (0, T), \\ y = 0, \text{ on } \Sigma_T = \partial\Omega \times (0, T), \\ y(0, \cdot) = y_0(\cdot), \text{ in } \Omega, \end{cases} \quad (1.0.1)$$

with $\theta = 0$ (resp. $\theta = \pi/2$) in the parabolic case (resp. for Schrödinger case). Here Ω is an open non-empty subset in \mathbb{R}^d with a smooth boundary Γ , $Y = (y_1, y_2)$ is the state to be controlled, \mathcal{C} is a coupling bounded operator on $(L^2(\Omega))^2$, B is either a bounded or unbounded control operator acting on a single component of the above system, and v is the scalar control. One can also consider the corresponding hyperbolic systems, obtained by replacing $e^{i\theta} y_t$ by y_{tt} (together with appropriate initial conditions).

The above systems have received a lot of attention in the case of cascade 2-coupled parabolic systems. An example of such cascade system arises when \mathcal{C}

$$\mathcal{C} = \begin{pmatrix} 0 & \mathbb{1}_O \\ 0 & 0 \end{pmatrix}, \quad (1.0.2)$$

with for instance $Bv = (0, v\mathbb{1}_\omega)^t$. Here O and ω are open non empty subsets of Ω standing respectively for the coupling and control regions and $\mathbb{1}_O$, the coupling coefficient, stands for the characteristic function of the set O . The coupling coefficient may be more generally a nonnegative (or nonpositive) function with a support that is included in Ω . One may also consider other types of coupling operators such as, for instance, symmetric coupling operators

$$\mathcal{C} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \quad (1.0.3)$$

where $c \geq 0$ on Ω . One can also consider more general coupling operators.

Some overview on the literature for controlled coupled systems

Let us first present some of the results for parabolic equations. De Teresa [47] considered 2-coupled cascade parabolic systems with a single locally distributed control. Under the assumption that $\omega \cap O \neq \emptyset$, she proved null controllability by a single control, and as a byproduct she obtained the existence of insensitizing controls for the scalar heat equation. In the case $\omega \cap O = \emptyset$, Kavian and de Teresa [29] proved a unique continuation result for a 2-coupled cascade systems of parabolic equations. De Teresa and Zuazua [48] give further results, concerning the determination of the initial data for which insensitizing controls of the heat equation can be built. Null controllability results by a reduced number of controls, for more general coupled parabolic systems

are proved mainly by means of Carleman estimates, and in two types of situations. The first one is devoted to constant (or time-dependent) coupling operators. The second one assumes locally distributed couplings, and locally distributed controls, but then with a non-empty intersection between the coupling and the control regions. We refer to [47, 8, 9, 23, 24, 32, 18, 40, 36] and to the survey paper [10] for such results and to the references therein. Let us further mention an interesting result by Coron, Guerrero and Rosier [18], which proves local null controllability results for nonlinearly coupled 2-systems of parabolic equations with a *nonlinear coupling* term. Such systems arise from applications to the control of chemical reaction-diffusion models. These nonlinear control results are based on the Coron's return method [17].

Hence it is a challenging issue to prove positive controllability results for coupled systems with a reduced number of controls, especially in situations for which the coupling and control regions do not meet. It seems that up to now the direct methods for coupled parabolic systems, mainly based on Carleman estimates, do not allow to deal with a coupling region that does not meet the control region, and this for boundary as well as locally distributed control problems. We shall see below that indirect methods, based for instance on controllability results for the corresponding hyperbolic system, may answer partially this issue.

Let us now present some results of the literature for coupled hyperbolic systems. We shall first consider symmetric coupled systems of wave equations (see [1, 2]), that is

$$\begin{cases} y_{1,tt} - \Delta y_1 + C y_2 = B v, & \text{in } Q_T = \Omega \times (0, T), \\ y_{2,tt} - \Delta y_2 + C^* y_1 = 0, & \text{in } Q_T = \Omega \times (0, T), \\ y_i = 0 \quad i = 1, 2, & \text{on } \Sigma_T = \partial\Omega \times (0, T), \\ y_i(0, \cdot) = y_i^0(\cdot) \quad i = 1, 2, & \text{in } \Omega. \end{cases} \quad (1.0.4)$$

Here only the first equation is controlled. One can also consider the corresponding abstract system in which the Dirichlet laplacian is replaced by a general self-adjoint coercive operator in a Hilbert space. In [2], we introduce an original method –named *two-level energy method*– to prove positive controllability results. This method is set in a general abstract setting and relies on several properties. We use the property of conservation of the total energy of the adjoint system, and a time-independent observability inequality for a scalar equation with a source term. Our method also relies on the idea to work in a weakened energy space for the unobserved component and to use a balance of energies between the unobserved and the observed components. Thanks to this, we prove observability and controllability results for coupled systems in the case of coercive bounded coupling operators C (case of globally distributed couplings), unbounded control operators (case of boundary control), and in some situations if the diffusion operators are not the same. The results of [1, 2] have been recently extended by the author and Léautaud in [3, 4] to the case of partially coercive coupling operators (case of localized couplings). The coupling coefficient is assumed to be a sufficiently smooth function. The results are valid for localized as well as boundary control, and for the same diffusion operators. The geometric assumptions are that the control and coupling regions should both satisfy the Geometric Control Condition of Bardos Lebeau Rauch [11] (see also [15, 16] for weaker smoothness assumptions on Ω and the coefficients of the elliptic operator A). This allows many situations for which the control and coupling regions do not necessarily meet.

We can now turn to some consequences for the corresponding parabolic or Schrödinger systems. The transmutation method [43, 39, 41, 21], allows to deduce null controllability results for heat or Schrödinger equations from exact controllability results for the wave equation. This is an indirect method. Thus, using the transmuta-

tion method, we prove in [3, 4], null controllability for symmetric 2-coupled systems of parabolic and diffusive equations. These results are valid in a multi-dimensional setting and under the condition that both the coupling and control regions satisfy the Geometric Control Condition. In particular, this allows many geometric situations for which these regions do not meet. A drawback of this indirect method is that these null controllability results are then, only valid under the usual geometric control conditions for the wave equations, whereas for instance null controllability for the scalar heat equation holds without geometric assumptions.

Let us now give an overview of the literature on insensitizing controls for the scalar wave equation and on controllability of 2-coupled hyperbolic cascade systems. As far as we know, the first results for insensitizing controls for the scalar wave equation are due to Dáger [19]. He proved the insensitizing boundary controllability, and the ε -insensitizing locally distributed controllability for the one-dimensional wave equation. In both situations, the control regions need not to meet the coupling region. Tebou [44] has considered the same questions in the multi-dimensional framework. He partially extended Dáger's results to the multi-dimensional wave equation for controls which are locally distributed in a region ω , and furthermore coupling regions O which necessarily meet ω . More precisely, he proved the ε -insensitizing locally distributed controllability for arbitrary open subsets ω and O such that $\omega \cap O \neq \emptyset$. He also proved the insensitizing locally distributed controllability under a strong geometric assumption, namely that both the control and coupling regions contain the same neighbourhood of a part Γ_1 of the boundary, that satisfies the usual multiplier condition. A slightly different analysis has been performed by Tebou in [45]. In this paper, the functional to be insensitized involves the trace of the normal derivative of the unknown on a part Γ_1 of the boundary. The existence of a locally distributed insensitizing control is proved under the strong geometric condition that Γ_1 satisfies the usual multiplier geometric condition and that the localized control region contains a neighbourhood of Γ_1 .

In a recent work, Rosier and de Teresa [42] considered a 2-coupled system of cascade hyperbolic equations under a strong hypothesis, that is a periodicity assumption of the semigroup associated to a single uncoupled equation. They give applications to 2-coupled systems of cascade one-dimensional heat equations and to 2-coupled systems of cascade Schrödinger equations in a n -dimensional interval with empty intersection between the control and coupling regions. The coupling coefficient is a characteristic function and thus is not a smooth function. Their method is linked to Dáger's [19] approach, based on the periodicity assumption of the semigroup for the single free equation. Dehman Léautaud Le Rousseau [33] (see also [20]) consider coupled cascade wave systems in a \mathcal{C}^∞ compact connected riemannian manifold without boundary in the case of locally distributed observations. Using a contradiction argument, they prove an observability inequality for such systems, and further give the characterization of the minimal control time. Their proof uses micro-local analysis and the principle to work in weakened energy spaces for the unobserved component (see [1, 2]). They deduce the corresponding controllability result thanks to the HUM method.

We also would like to refer to some books on coupled systems and on control theory. In particular, the interested reader can find a series of results on the control of coupled systems issued from mechanical applications, and in particular for acoustic models, in Lasiecka [31]. We refer to J.-L. Lions [34] for the HUM method, together with examples of partial control of coupled systems in the case of infinite dimensional control systems and to Coron [17] for a nice introduction of the HUM method in a finite dimensional setting, a general approach of nonlinear control and various examples of nonlinear control for PDE's.

Main results of the paper

We give a necessary and sufficient condition for the observability of 2-coupled abstract hyperbolic cascade systems by a single observation. The observability operator can be bounded as well as an unbounded operator. The coupling operator is assumed to be only partially coercive as in [4]. In the case of domains such that their boundaries have no contact of infinite order with its tangent (or for analytic boundaries), this necessary and sufficient condition states that the support of the observation and coupling functions should satisfy (GCC). In the one-dimensional case, this shows that it is possible to drive back to equilibrium the solution of an uncontrolled wave equation locally coupled to a controlled scalar wave equation, the coupling being active on any non-empty open subset of arbitrary measure. We prove these results, by adapting the two-level energy method [1, 2, 4] to cascade systems. This method has been originally introduced to handle symmetric coupled systems for which the total energy of the solutions is conserved through time, whereas this property is lost for cascade systems. We deduce controllability results for coupled wave systems. Using transmutation, we prove null-controllability results for coupled heat or Schrödinger systems under geometric conditions. These conditions are those of hyperbolic systems. We give several applications of these results to 2-coupled cascade wave, heat and Schrödinger systems. They can also be applied to higher order systems such as Petrowsky equations, or wave equations with variable coefficients. The main point of this paper is that these results are valid in a multi-dimensional framework, for locally distributed as well as boundary controls (resp. observations), and for localized couplings in situations for which the control/observations regions do not meet the localized coupling regions. They answer positively (and partially due to geometric conditions) to an open question raised by Luz de Teresa [47] for forward cascade heat systems.

We further give applications to the question of existence of exact insensitizing locally distributed and boundary controls for the wave equation as introduced by J.-L. Lions [35], obtaining new and complete results on this question, generalizing those of Dáger [19] and Tebou [44] (see also [45]). This result is important since it allows to build exact controls for the scalar wave equation which are robust, since these controls both drive back the solution to equilibrium and insensitize a weighted observation of the solution, making this observation insensitive to small unknown perturbations of the initial data.

Parts of these results (without proofs) have been announced in [5]. It should be noted that it is not possible for insensitizing control of the heat equation to use results from coupled cascade wave systems through the transmutation method. This comes from the fact the 2-coupled cascade systems issued from insensitizing control for the heat equation, are two coupled heat equations one being forward in time, the second being backward in time.

2 Controllability and observability of 2-coupled cascade hyperbolic systems by a single control/observation

2.1 Observability of 2-coupled cascade hyperbolic systems by a single observation

We consider the following coupled cascade hyperbolic system of order 2

$$\begin{cases} u_1'' + Au_1 = 0, \\ u_2'' + Au_2 + C_{21}u_1 = 0, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2, \end{cases} \quad (2.1.1)$$

where H is an Hilbert space with norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$ and C_{21} is a bounded operator in H . We assume that A satisfies

$$(A1) \quad \begin{cases} A : D(A) \subset H \mapsto H, A^* = A, \\ \exists \omega > 0, |Au| \geq \omega|u| \quad \forall u \in D(A), \\ A \text{ has a compact resolvent.} \end{cases} \quad (2.1.2)$$

One can note that this system is a lower triangular system, that is it involves a lower triangular operator when written as a second order equation in vectorial form. More precisely, we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}'' = \begin{pmatrix} -A & 0 \\ -C_{21} & -A \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

We set $H_k = D(A^{k/2})$ for $k \in \mathbb{N}$, with the convention $H_0 = H$. The set H_k is equipped with the norm $|\cdot|_k$ defined by $|A^{k/2} \cdot|$ and the associated scalar product. It is a Hilbert space. We denote by H_{-k} the dual space of H_k with the pivot space H . We equip H_{-k} with the norm $|\cdot|_{-k} = |A^{-k/2} \cdot|$. We define the energy space associated to (2.1.1) by $\mathcal{H} = H_1^2 \times H_0^2$.

The system (2.1.1) can then be reformulated as the first order abstract system

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = U^0 = (u_1^0, u_2^0, u_1^1, u_2^1), \end{cases} \quad (2.1.3)$$

where $U = (u_1, u_2, v_1, v_2)$ and \mathcal{A} is the unbounded operator in \mathcal{H} with domain $D(\mathcal{A}) = H_2^2 \times H_1^2$ defined by

$$\mathcal{A}U = (v_1, v_2, -Au_1, -Au_2 - C_{21}u_1). \quad (2.1.4)$$

Using semigroup theory, it is easy to establish the well-posedness of the abstract system (2.1.3) for initial data $U_0 \in \mathcal{H}$. Moreover for initial data $U^0 \in D(\mathcal{A})$, we easily deduce that $Cu_1 \in \mathcal{C}^1([0, T]; H)$. Using classical results on inhomogeneous Cauchy problems for the first order equation satisfied by $U_2 = (u_2, u_2')$ and the regularity of Cu_1 , one can check that for $U^0 \in D(\mathcal{A})$, the solution U of (2.1.3) is such that $U \in \mathcal{C}([0, T]; D(\mathcal{A})) \cap \mathcal{C}^1([0, T]; \mathcal{H})$. Moreover, assuming that $C_{21} \in \mathcal{L}(H_{k-1})$ for $k \in \mathbb{Z}^*$, the problem (2.1.3) (and similarly (2.1.1)) is well-posed in $H_k^2 \times H_{k-1}^2$, that is if the initial data are in $H_k^2 \times H_{k-1}^2$, then the solution U of (2.1.3) (and similarly that of (2.1.1)) is in $\mathcal{C}([0, T]; H_k^2 \times H_{k-1}^2)$. For a solution $U = (u_1, u_2, v_1, v_2)$ of (2.1.1), we have $v_i = u_i'$ for $i = 1, 2$. In the sequel, we will need several levels of energy of solutions U of (2.1.3). For this, it is convenient to introduce some further notation. For a solution $U = (u_1, u_2, u_1', u_2')$ of (2.1.3), we set

$$U_i = (u_i, u_i') \text{ for } i = 1, 2. \quad (2.1.5)$$

For $U_i \in H_k \times H_{k-1}$, we define the *local* energies of level k as

$$e_k(U_i)(t) = \frac{1}{2} \left(|A^{k/2} u_i|^2 + |A^{(k-1)/2} u_i'|^2 \right), \quad k \in \mathbb{Z}, i = 1, 2. \quad (2.1.6)$$

For $k = 1$, we recover the natural energy of each component of the state. For $k < 1$, these energies are weakened energies. We will also use this notation for more general set of vector-valued functions $t \mapsto V(t) = (v_1(t), v_2(t)) \in H_k \times H_{k-1}$ for convenience, that is we define $e_k(V)(t)$ as above without further recalling this in the sequel.

We will also need *global* energies of level k . For this, we need to invert \mathcal{A} on the set of solutions $U = (u_1, u_2, v_1, v_2)$ of (2.1.3). We check that this is possible in the following proposition.

Proposition 2.1. *Assume that A satisfies (A1) and define \mathcal{A} as in (2.1.4). Then \mathcal{A} is invertible from $D(\mathcal{A})$ on \mathcal{H} . Moreover for any solution $U = (u_1, u_2, v_1, v_2)$ of (2.1.3), the equation*

$$\mathcal{A}W = U, \quad (2.1.7)$$

admits a unique solution $W = (w_1, w_2, r_1, r_2)$ given by

$$\begin{cases} w_1 = -A^{-1}u'_1, \\ w_2 = -A^{-1}u'_2 + A^{-1}C_{21}A^{-1}u'_1, \\ r_1 = w'_1 = u_1, r_2 = w'_2 = u_2. \end{cases} \quad (2.1.8)$$

Also, W is then the solution of (2.1.3), associated to the initial data $W(0) = (A^{-1}u_1^1, -A^{-1}u_2^1 + A^{-1}C_{21}A^{-1}u_1^1, u_1^0, u_2^0)$.

Proof. We associate to $U = (u_1, u_2, v_1, v_2) \in \mathcal{H}$, the vector $W = (w_1, w_2, r_1, r_2)$ defined by

$$\begin{cases} w_1 = -A^{-1}v_1, \\ w_2 = -A^{-1}v_2 + A^{-1}C_{21}A^{-1}v_1, \\ r_1 = u_1, r_2 = u_2. \end{cases} \quad (2.1.9)$$

One can check that $W \in D(\mathcal{A})$ and satisfies (2.1.7). Hence \mathcal{A} is invertible. On the other hand U satisfies (2.1.3). Thus applying the operator \mathcal{A}^{-1} on both sides of the first equation of (2.1.3), we deduce that W also satisfies

$$W' = \mathcal{A}W.$$

Therefore, by definition of \mathcal{A} , we have $w'_i = r_i$ for $i = 1, 2$. In a similar way, we have $u'_i = v_i$ for $i = 1, 2$. This, together with (2.1.9) imply that W is the unique solution of (2.1.8). Moreover if U is a solution of (2.1.3), then W is the solution of (2.1.3) corresponding to initial data $W(0) = (A^{-1}u_1^1, -A^{-1}u_2^1 + A^{-1}C_{21}A^{-1}u_1^1, u_1^0, u_2^0)$, and given by (2.1.8). \square

We have the following corollary, which proof is left to the reader.

Corollary 2.2. *We assume the hypotheses of Proposition 2.1. Then the equation*

$$\mathcal{A}^k W^k = U \quad (2.1.10)$$

admits a unique solution $W^k \in D(\mathcal{A}^k)$ defined by induction as

$$W^0 = U, W^{k+1} = \mathcal{A}^{-1}W^k, k \in \mathbb{N}. \quad (2.1.11)$$

Remark 2.3. *In the above corollary, we can only assert that W^k is in $D(\mathcal{A}^k)$. If in addition, the bounded operator C_{21} satisfies $C_{21}H_{k-1} \subset H_{k-1}$ for $k \in \mathbb{N}^*$, then $W^k \in H_{k+1}^2 \times H_k^2$.*

2.1.1 Main results for the observability of 2-coupled cascade systems

Our purpose is to establish indirect observability estimates for the system (2.1.1). For this, we shall assume the following hypotheses (A2) – (A5).

We assume that the coupling operator C_{21} satisfies

$$(A2) \begin{cases} C_{21}^* \in \mathcal{L}(H_k) \text{ for } k \in \{0, 1\}, \\ \|C_{21}\| = \beta, |C_{21}w|^2 \leq \beta \langle C_{21}w, w \rangle \quad \forall w \in H, \\ \exists \alpha > 0 \text{ such that, } \alpha |\Pi w|^2 \leq \langle C_{21}w, w \rangle \quad \forall w \in H, \end{cases} \quad (2.1.12)$$

where the operator Π satisfies the assumptions

$$(A3) \begin{cases} \Pi \in \mathcal{L}(H), \exists T_0 > 0, \forall T > T_0, \exists C_2(T) > 0 \text{ such that} \\ \forall (w^0, w^1) \in H_1 \times H \text{ the solution } w \text{ of} \\ w'' + Aw = 0, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T |\Pi w'|^2 dt \geq C_2(T) e_1(W)(0), \end{cases} \quad (2.1.13)$$

where $W = (w, w')$. We denote by G a given Hilbert space with norm $\|\cdot\|_G$ and scalar product $\langle \cdot, \cdot \rangle_G$. The space G will be identified to its dual space in all the sequel. We make the following assumptions on the observability operator \mathbf{B}^* (the dual operator of the control operator \mathbf{B}).

We shall first assume that \mathbf{B}^* is an admissible observation operator for one equation, that is

$$(A4) \begin{cases} \mathbf{B}^* \in \mathcal{L}(H_2 \times H; G), \\ \forall T > 0 \exists C > 0, \text{ such that for all } (w^0, w^1) \in H_1 \times H \text{ and all } f \in L^2([0, T]; H), \\ \text{the solution } w \text{ of } w'' + Aw = f, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \leq C \left(e_1(W)(0) + e_1(W)(T) + \int_0^T e_1(W)(t) dt + \int_0^T |f|^2 dt \right), \end{cases} \quad (2.1.14)$$

where $W = (w, w')$.

We further assume the following observability inequality for a single equation

$$(A5) \begin{cases} \exists T_0 > 0, \forall T > T_0, \exists C_1(T) > 0 \text{ such that} \\ \forall (w^0, w^1) \in H_1 \times H, \text{ the solution } w \text{ of} \\ w'' + Aw = 0, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \geq C_1(T) e_1(W)(0). \end{cases} \quad (2.1.15)$$

Remark 2.4. *The minimal times for which the two observability inequalities hold in (A3) and (A5) are not necessarily the same for the two observability operators, but here, we only consider the sup of these two optimal times to avoid too many notation.*

Lemma 2.5. (*Admissibility property*) Assume the hypotheses (A1), (A4) and that $C_{21} \in \mathcal{L}(H)$, then for all $T > 0$, there exists a constant $C = C(T) > 0$ such that for all initial data $U^0 \in \mathcal{H}$, the solution of (2.1.1) satisfies the following direct inequality

$$\int_0^T \|\mathbf{B}^* U_2\|_G^2 dt \leq C \left(e_0(U_1)(0) + e_1(U_2)(0) \right). \quad (2.1.16)$$

Remark 2.6. This Lemma establishes a hidden regularity property of the solutions, namely that for all $U^0 \in \mathcal{H}$, $\mathbf{B}^* U_2 \in L^2([0, T]; G)$.

Theorem 2.7 (Sufficient conditions). Assume the hypotheses (A1) – (A5). Then there exists $T_3 > 0$ such that for all $T > T_3$, and all initial data $U^0 \in \mathcal{H}$, the solution of (2.1.1) satisfies the observability estimates

$$\begin{cases} d_1(T) \int_0^T \|\mathbf{B}^* U_2\|_G^2 \geq e_0(U_1)(0), \\ d_2(T) \int_0^T \|\mathbf{B}^* U_2\|_G^2 \geq e_1(U_2)(0), \end{cases} \quad (2.1.17)$$

where the constants $d_i(T) > 0$ are obtained thanks to (2.1.45) (in Lemma 2.19) and (2.1.42) (in Lemma 2.18), depend on T and satisfy for T sufficiently large

$$d_1(T) \leq \frac{K}{T^3}, \quad d_2(T) \leq \frac{K}{T}. \quad (2.1.18)$$

Moreover, the following estimates also hold

$$\int_0^T e_1(U_2) \leq k_2(T) \int_0^T \|\mathbf{B}^* U_2\|_G^2, \quad (2.1.19)$$

and

$$\int_0^T \langle C_{21} u_1, u_1 \rangle \leq r_2(T) \int_0^T \|\mathbf{B}^* U_2\|_G^2, \quad (2.1.20)$$

where for T sufficiently large

$$k_2(T) \leq K, \quad r_2(T) \leq K/T^2. \quad (2.1.21)$$

Here the notation K stands for positive constants which does not depend on, T , but depends on α, β, γ_0 in an explicit way. Moreover, one can choose T_3 in the following form $T_3 = \max(T_0, T_1, T_2)$, where T_0 is introduced in (A3), (A5) and T_1, T_2 are defined in (2.1.41). In addition, if C_{21} satisfies (A3), then one can choose $\Pi = C_{21}$ so that the above properties hold.

We also prove that the above conditions are optimal in the following theorem.

Theorem 2.8 (Necessary conditions). Assume the hypotheses (A1) and (A4), and that

$$(A2)' \begin{cases} C_{21}^* \in \mathcal{L}(H_k) \text{ for } k \in \{0, 1\}, \\ \|C_{21}\| = \beta, |C_{21}w|^2 \leq \beta \langle C_{21}w, w \rangle \quad \forall w \in H, \end{cases} \quad (2.1.22)$$

holds. Assume that $\Pi = C_{21}$ does not satisfy (A3) or that \mathbf{B} does not satisfy (A5). Then there does not exist $T_3 > 0$ such that for all $T > T_3$, the following property holds

$$(OBS) \begin{cases} \exists C > 0 \text{ such that } \forall U^0 \in \mathcal{H} \text{ the solution of (2.1.1) satisfies} \\ C(e_0(U_1)(0) + e_1(U_2)(0)) \leq C \int_0^T \|\mathbf{B}^* U_2\|_G^2 dt. \end{cases} \quad (2.1.23)$$

Corollary 2.9. Assume (A1) and (A4) and that $\Pi = C_{21}$ satisfies (A2)'. Then (2.1.23) holds if and only if (A3) and (A5) hold.

2.1.2 Proofs of the main results for the observability of 2-coupled systems

Proof. of Lemma 2.5. Thanks to assumption (A4) applied to the second equation of (2.1.1), we have for all $T > 0$ there exists $C(T) > 0$ such that

$$\int_0^T \|\mathbf{B}^* U_2\|_G^2 dt \leq C(T) \left(e_1(U_2)(0) + e_1(U_2)(T) + \int_0^T e_1(U_2)(t) dt + \int_0^T |C_{21} u_1|^2 dt \right). \quad (2.1.24)$$

On the other hand, the usual energy estimates yield

$$e_1(U_2)(t) \leq C \left(e_1(U_2)(0) + T \int_0^T |C_{21} u_1|^2 dt \right).$$

Integrating this inequality between 0 and T and using it for $t = T$ in (2.1.24), we obtain (2.1.16). \square

The proof of Theorem 2.7 follows the idea of the two-level energy method such as developed in [2]. It consists in using two level of energies, the natural one for the observed component of the state and the weakened energy of the unobserved component of the state. Here, we no longer consider symmetric conservative coupled hyperbolic systems such as considered in [2], or more recently in [3]. For the two-level energy method in the situation of [2], a crucial property was assumed for the abstract system (and proved for the applicative examples), that is direct and observability inequalities for a single equation with a source term, with constants which are *uniform* with respect to the length T of the time interval $[0, T]$. This property was proved thanks to the multiplier method for the usual PDE's (wave, Petrowsky, ...) in [2] and extended in [3] under an abstract form which allows the use of the optimal geometric conditions of Bardos Lebeau and Rauch [11]. We shall use this result in the sequel so we recall that it reads as follows.

Lemma 2.10 ([4], [Lemma 3.3, pp. 14]). *We assume the hypotheses (A1), (A4) and (A5). Then, there exist constants $\eta_0 > 0$ and $\alpha_0 > 0$ such that for all $T > T_0$, and for any solution $P = (p, p')$ of the nonhomogeneous equation*

$$p'' + Ap = f \in L^2([0, T]; H), \quad (2.1.25)$$

the following uniform observability estimate holds

$$\eta_0 \int_0^T \|\mathbf{B}^* P\|^2 dt \geq \int_0^T e_1(P)(t) dt - \alpha_0 \int_0^T |f|^2 dt. \quad (2.1.26)$$

In a similar way if (A1) and (A3) hold then there exist $\gamma_0 > 0$ and $\delta_0 > 0$ such that for all $T > T_0$, and for any solution $P = (p, p')$ of (2.1.25), the following uniform observability estimate holds

$$\gamma_0 \int_0^T |\Pi p'|^2 dt \geq \int_0^T e_1(P)(t) dt - \delta_0 \int_0^T |f|^2 dt. \quad (2.1.27)$$

Remark 2.11. Note that we give a slightly different presentation and use different notations for the constants than in the original Lemma 3.3 in [4] without loss of generality.

We deduce from this lemma the following corollary.

Corollary 2.12. Assume that (A1) and (A3) hold, then there exists a constant $\gamma_0 > 0$ such that for all $T > T_0$ and for all solutions $V = (v, v')$ of

$$\begin{cases} v'' + Av = 0, \\ (v, v')(0) = (v^0, v^1), \end{cases} \quad (2.1.28)$$

the following observability inequality holds

$$Te_1(V)(0) \leq \gamma_0 \int_0^T |\Pi v'|^2 dt. \quad (2.1.29)$$

Proof. We use the inequality (2.1.27) with $f = 0$ and the conservation of energy. This gives the desired result. \square

Remark 2.13. It should be noted that in (A3), the dependence of the constant $C_2(T)$ on T is not known. Here we can prove that this constant can be chosen under the form $C_2(T) = T/\gamma_0$. This is an important property for the two-level energy method.

Proof of Theorem 2.7:

The proof will be divided in several Lemma.

In the sequel, we will assume that the initial data for (2.1.3) are in $D(\mathcal{A})$. The final result for all initial data follows easily from the density of $D(\mathcal{A})$ in \mathcal{H} . We will also use the above notation without further specifying it. Furthermore, we set $C = C_{21}$ in the sequel of this subsection.

As in [2], we obtain a first estimate using the coupling operator and involving the natural energy of u_2 and the weakened energy of u_1 .

Lemma 2.14. We assume (A1). Let $U^0 \in D(\mathcal{A})$. Then the solution of (2.1.1) satisfies the estimate

$$\int_0^T \langle Cu_1, u_1 \rangle \leq 2\eta e_0(U_1)(0) + \frac{1}{\eta} (e_1(U_2)(T) + e_1(U_2)(0)), \forall \eta > 0. \quad (2.1.30)$$

Proof. Since (u_1, u_2) is a solution of (2.1.1), we have

$$\int_0^T \langle u_1'' + Au_1, u_2 \rangle - \langle u_2'' + Au_2 + Cu_1, u_1 \rangle = 0,$$

so that

$$\int_0^T \langle Cu_1, u_1 \rangle = \left[\langle u'_1, u_2 \rangle - \langle u'_2, u_1 \rangle \right]_0^T. \quad (2.1.31)$$

We estimate the right hand side of (2.1.31) as follows. We set

$$I(t) = \langle u'_1, u_2 \rangle - \langle u'_2, u_1 \rangle.$$

Then we have

$$|I(t)| \leq \frac{\eta}{2} \left(|A^{-1/2} u'_1|^2 + |u_1|^2 \right) + \frac{1}{2\eta} \left(|A^{1/2} u_2|^2 + |u'_2|^2 \right).$$

Since the weakened energy $e_0(U_1)$ is conserved, we obtain the desired estimate. \square

Corollary 2.15. *We assume (A1) – (A5). Let $U^0 \in D(\mathcal{A})$. Then the solution of (2.1.1) satisfies*

$$\int_0^T \langle Cu_1, u_1 \rangle \leq \frac{8\gamma_0}{\alpha T} (e_1(U_2)(T) + e_1(U_2)(0)). \quad (2.1.32)$$

Proof. We define $W = (w_1, w_2, r_1, r_2)$ by (2.1.7). Then we proved in Proposition 2.1 that W solves (2.1.8) and (2.1.3), so that in particular (w_1, w'_1) satisfies (2.1.28). Thanks to the hypothesis (A3) for (w_1, w'_1) and to Lemma 2.10, (2.1.29) holds for $V = W_1$. We note that $w'_1 = u_1$. Thus, thanks to the hypotheses (A2) – (A3) and to (2.1.29) together with (2.1.30) with the choice $\eta = \frac{T\alpha}{4\gamma_0}$, we obtain

$$e_0(U_1)(0) \leq \frac{8\gamma_0^2}{\alpha^2 T^2} (e_1(U_2)(T) + e_1(U_2)(0)).$$

Using this estimate in (2.1.30) with the above choice of η , we get (2.1.32). \square

Lemma 2.16. *Assume the hypotheses of Corollary (2.15). Then,*

$$(e_1(U_2)(T) + e_1(U_2)(0)) \leq c_1 e_1(U_2)(0) + \frac{c_2 \beta \gamma_0}{\alpha T} \int_0^T e_1(U_2), \quad (2.1.33)$$

and,

$$\int_0^T \langle Cu_1, u_1 \rangle \leq \frac{c_3 \gamma_0}{\alpha T} e_1(U_2)(0) + \frac{c_4 \beta \gamma_0^2}{\alpha^2 T^2} \int_0^T e_1(U_2). \quad (2.1.34)$$

Proof. Let (u_1, u_2) be a solution of (2.1.1), then we have

$$\langle u_2'' + Au_2 + Cu_1, u_2' \rangle = 0,$$

so that

$$e_1'(U_2)(t) = -\langle Cu_1, u_2' \rangle(t). \quad (2.1.35)$$

Integrating this relation between 0 and T , we obtain

$$e_1(U_2)(T) + e_1(U_2)(0) \leq 2e_1(U_2)(0) + \frac{\alpha T}{16\gamma_0} \int_0^T \langle Cu_1, u_1 \rangle + \frac{8\beta\gamma_0}{\alpha T} \int_0^T e_1(U_2).$$

We use (2.1.32) in this last estimate. This gives (2.1.33). Using (2.1.33) in (2.1.32), we obtain (2.1.34). \square

Lemma 2.17. *Assume the hypotheses of Corollary 2.15. Then,*

$$\int_0^T e_1(U_2)(t) \geq M T e_1(U_2)(0), \quad (2.1.36)$$

where M is defined by (2.1.40) and depends only on α, β, γ_0 .

Proof. We set

$$a = \frac{c_3\beta\gamma_0}{2\alpha}, \quad (2.1.37)$$

and

$$b = \frac{c_4\beta^2\gamma_0^2}{2\alpha^2}. \quad (2.1.38)$$

We define

$$\nu = (a + \sqrt{a^2 + a + b}) \quad (2.1.39)$$

We integrate twice the two sides of (2.1.35), first between 0 and s and then between 0 and T . This gives

$$\int_0^T e_1(U_2)(t) = T e_1(U_2)(0) - \int_0^T (T - t) \langle Cu_1, u_2' \rangle(t).$$

Using Young's inequality on the second term of the right hand side in the above relation, together with (2.1.12) in assumption (A2) and the definition of $e_1(U_2)$, we deduce that

$$(1 + \nu) \int_0^T e_1(U_2)(t) \geq T e_1(U_2)(0) - \frac{\beta T^2}{2\nu} \int_0^T \langle Cu_1, u_1 \rangle.$$

Using (2.1.34) in this last estimate and the definition of a, b and ν , we obtain (2.1.36) where $M = M(\alpha, \beta, \gamma_0)$ is defined by

$$M = \frac{\sqrt{a^2 + a + b}}{(2a + 1)(a + \sqrt{a^2 + a + b}) + a + 2b}. \quad (2.1.40)$$

\square

Lemma 2.18. *Assume the hypotheses of Theorem 2.7. We set*

$$T_1 = \frac{\sqrt{2c_4\alpha_0\beta\gamma_0}}{\alpha}, T_2 = \frac{\sqrt{2c_3\alpha_0\beta\gamma_0}}{\sqrt{\alpha M}}, T_3 = \max(T_0, T_1, T_2), \quad (2.1.41)$$

where T_0 is introduced in (A3) and (A5). Then for all $T > T_3$, we have

$$\eta_0 \int_0^T \|\mathbf{B}^* U_2\|_G^2 \geq \frac{M}{2T} (T^2 - T_2^2) e_1(U_2)(0), \quad (2.1.42)$$

and

$$\int_0^T e_1(U_2) \leq 2\eta_0 \frac{T^2}{T^2 - T_2^2} \int_0^T \|\mathbf{B}^* U_2\|_G^2. \quad (2.1.43)$$

Proof. Applying Lemma 2.10 for the equation satisfied by u_2 in (2.1.1), we deduce that (2.1.26) holds, that is

$$\eta_0 \int_0^T \|\mathbf{B}^* U_2\|_G^2 \geq \int_0^T e_1(U_2)(t) - \alpha_0 \beta \int_0^T \langle C u_1, u_1 \rangle. \quad (2.1.44)$$

We use (2.1.34) and (2.1.36) in this last inequality. Then for all $T > T_3$, we obtain (2.1.42). Using in a similar way, (2.1.34) and (2.1.36) in (2.1.44), together with the definitions of T_1 and T_2 , and (2.1.42) in the resulting equation, we obtain (2.1.43). \square

Hence, we proved that we can reconstruct the initial data of the second component of the state, which is coupled to the first component, from the observation of this second component. We now have to prove that we can reconstruct the weakened energy of the first component from the observation of the second component.

Lemma 2.19. *Assume the hypotheses of Theorem 2.7. We define T_3 as in (2.1.41). Then for all $T > T_3$, we have*

$$e_0(U_1)(0) \leq \frac{2\eta_0\gamma_0^2}{\alpha^2(T^2 - T_2^2)T} \left[\frac{c_3}{M} + \frac{c_4\beta\gamma_0}{\alpha} \right] \int_0^T \|\mathbf{B}^* U_2\|_G^2. \quad (2.1.45)$$

Proof. Using (2.1.44) and (2.1.34), and since $T > T_3$, we obtain

$$\int_0^T \langle C u_1, u_1 \rangle \leq \left(\frac{c_3\gamma_0}{\alpha T} \right) e_1(U_2)(0) + \frac{2c_4\eta_0\beta\gamma_0^2}{\alpha^2(T^2 - T_2^2)} \int_0^T \|\mathbf{B}^* U_2\|_G^2. \quad (2.1.46)$$

We define $W = (w_1, w_2, r_1, r_2)$ by (2.1.7). Then we proved in Proposition 2.1 that W solves (2.1.8) and (2.1.3), so that in particular (w_1, w_1') satisfies (2.1.28). Hence thanks to the uniform observability inequality (2.1.29), and to (A2), we deduce that

$$e_0(U_1)(0) \leq \frac{\gamma_0}{\alpha T} \int_0^T \langle C u_1, u_1 \rangle. \quad (2.1.47)$$

Inserting (2.1.42) in (2.1.46) and using (2.1.47) we derive (2.1.45). \square

Thus, thanks to (2.1.42) and (2.1.45), we establish indirect observability for the triangular hyperbolic system (2.1.1). We also deduce easily the estimates (2.1.19) and (2.1.20), so that Theorem 2.7 is proved.

Proof. of Theorem 2.8. Assume first that \mathbf{B}^* does not satisfy (A5). We argue by contradiction and assume that there exists $T_3 > 0$ such that (OBS) holds for all $T > T_3$. Then choosing the initial data U^0 under the form $U^0 = (0, u_2^0, 0, u_2^1)$ where (u_2^0, u_2^1) is arbitrary in $H_1 \times H_0$ we have by uniqueness that $u_1 = u_1' \equiv 0$ and u_2 is the solution of

$$\begin{cases} u_2'' + Au_2 = 0, \\ (u_2, u_2')(0) = (u_2^0, u_2^1), \end{cases} \quad (2.1.48)$$

whereas (OBS) reduces to: there exists $T_3 > 0$ such that for all $T > T_3$ we have

$$\begin{cases} \exists C > 0 \text{ such that } \forall (u_2^0, u_2^1) \in H_1 \times H_0 \text{ the solution of (2.1.48) satisfies} \\ C(e_1(U_2)(0)) \leq C \int_0^T \|\mathbf{B}^* U_2\|_G^2 dt. \end{cases}$$

Hence \mathbf{B}^* satisfies (A5) which contradicts our hypothesis.

Assume now that $\Pi = C_{21}$ does not satisfy (A3). We argue again by contradiction and assume that there exists $T_3 > 0$ such that (OBS) holds for all $T > T_3$. We now choose the initial data U^0 under the form $U^0 = (u_1^0, 0, u_1^1, 0)$ where (u_1^0, u_1^1) is arbitrary in $H_1 \times H_0$. Then (OBS) reduces to: there exists $T_3 > 0$ such that for all $T > T_3$ we have

$$\begin{cases} \exists C_1 > 0 \text{ such that } \forall (u_1^0, u_1^1) \in H_1 \times H_0 \text{ the solution of (2.1.1) satisfies} \\ C_1 e_0(U_1)(0) \leq \int_0^T \|\mathbf{B}^* U_2\|_G^2 dt. \end{cases}$$

On the other hand using the admissibility assumption (A4) together with the usual energy estimates and the fact that $e_1(U_2)(0) = 0$ we have

$$\int_0^T \|\mathbf{B}^* U_2\|_G^2 dt \leq C \int_0^T |C_{21} u_1|^2 dt.$$

Hence there exists $C_2 > 0$ such that

$$C_2 e_0(U_1)(0) \leq \int_0^T |C_{21} u_1|^2 dt \quad \forall (u_1^0, u_1^1) \in H_1 \times H_0.$$

We set $w = -A^{-1}u_1'$. Then we have $w' = u_1$ and $w'' + Aw = 0$. We set $W = (w, w')$. Then we have $(w, w')(0) = (w^0, w^1) = (-A^{-1}u_1^1, u_1^0) \in H_2 \times H_1$. Then, thanks to the above inequality, we have for all $T > T_3$

$$\int_0^T |C_{21} w'|^2 \geq C_2 e_1(W)(0) \quad \text{for all the solutions of } w'' + Aw = 0, \quad (2.1.49)$$

with initial data $(w^0, w^1) \in H_2 \times H_1$. By density of $H_2 \times H_1$ in $H_1 \times H_0$, and continuity arguments, we deduce that (2.1.49) holds for any w solution of $w'' + Aw = 0$ with initial data $(w^0, w^1) \in H_1 \times H_0$, so that C_{21} satisfies (A3), which contradicts our hypothesis. \square

To handle the control problem, we shall need to prove the admissibility and observability properties under a slightly different form (mainly for the case $B \in \mathcal{L}(G, H)$). We have the following results.

Lemma 2.20. *Assume the hypotheses of Proposition 2.1. Then there exist $C > 0, C_1 > 0, C_2 > 0$ such that for all $W \in \mathcal{H}$, the following properties hold for $Z = \mathcal{A}^{-1}W$*

- (i) $e_0(Z_1) = e_{-1}(W_1)$,
- (ii) $e_0(W_2) \leq C(e_0(Z_1) + e_1(Z_2))$,
- (iii) $e_1(Z_2) \leq C(e_{-1}(W_1) + e_0(W_2))$,
- (iv) $C_1(e_0(Z_1) + e_1(Z_2)) \leq e_{-1}(W_1) + e_0(W_2) \leq C_2(e_0(Z_1) + e_1(Z_2))$.

Proof. $Z = (z_1, z_2, z'_1, z'_2)$ is defined as $z_1 = -A^{-1}w'_1$, $z_2 = -A^{-1}w'_2 + A^{-1}C_{21}A^{-1}w'_1$, $z'_i = w_i$ for $i = 1, 2$. Therefore (i) holds. On the other hand, we have

$$e_0(W_2) = \frac{1}{2}(|w_2|^2 + |A^{-1/2}w'_2|^2) \leq C(e_0(Z_1) + e_1(Z_2)).$$

We also have

$$e_1(Z_2) = \frac{1}{2}(|A^{1/2}z_2|^2 + |z'_2|^2) \leq C(e_{-1}(W_1) + e_0(W_2)).$$

We deduce easily (iv) thanks to (i) – (ii) – (iii). □

Remark 2.21. *The operator \mathcal{A} defined in (2.1.4) generates a \mathcal{C}^0 -semigroup on $H_{-1}^2 \times H_{-2}^2$. Hence due to the property of reversibility of time, the Cauchy problem $U' = \mathcal{A}U$, $U(T) = U^T \in H_{-1}^2 \times H_{-2}^2$ is well-posed, that is has a unique solution in $\mathcal{C}^0([0, T]; H_{-1}^2 \times H_{-2}^2)$. For the duality with the control problem, we shall need to work in different functional spaces which depend on the assumption on the control operator (case of bounded or unbounded control operator).*

We set

$$X_{-1} = H_{-1} \times H \times H_{-2} \times H_{-1}. \quad (2.1.50)$$

Since $X_{-1} \subset H_{-1}^2 \times H_{-2}^2$, we can also solve the Cauchy problem with $U^T \in X_{-1}$. In a similar way, we set

$$X_1 = H \times H_1 \times H_{-1} \times H. \quad (2.1.51)$$

Since $X_1 \subset H_{-1}^2 \times H_{-2}^2$, we can also solve the Cauchy problem with $U^T \in X_1$.

Lemma 2.22. *Assume (A1) – (A5). Let $T > 0$ be given. For $W^T = (w_1^T, w_2^T, q_1^T, q_2^T) \in X_{-1}$, we denote by $W = (w_1, w_2, w'_1, w'_2)$ the unique solution in $\mathcal{C}^0([0, T]; H_{-1}^2 \times H_{-2}^2)$ of*

$$\begin{cases} w_1'' + Aw_1 = 0, \\ w_2'' + Aw_2 + C_{21}w_1 = 0, \\ W|_{t=T} = W^T. \end{cases} \quad (2.1.52)$$

Then W satisfies the following properties

- (i) $W \in \mathcal{C}^0([0, T]; X_{-1})$,
- (ii) There exists $C_1 = C_1(T) > 0$, such that

$$C_1 \int_0^T \|\mathbf{B}^* Z_2\|_G^2 dt \leq e_{-1}(W_1)(0) + e_0(W_2)(0), \quad (2.1.53)$$

where $Z = \mathcal{A}^{-1}W$.

- (iii) For all $T > T_3$, where T_3 is given in Theorem 2.7, there exists $C_2 = C_2(T) > 0$ such that

$$e_{-1}(W_1)(0) \leq C_2 \int_0^T \|\mathbf{B}^* Z_2\|_G^2 dt, \quad (2.1.54)$$

$$e_0(W_2)(0) \leq C_2 \int_0^T \|\mathbf{B}^* Z_2\|_G^2 dt. \quad (2.1.55)$$

- (iv) Assume furthermore that $\mathbf{B}^*(w, w') = B^*w'$ where $B \in \mathcal{L}(G, H)$ is such that (A4) and (A5) hold. Then properties (ii) – (iii) become

$$C_1 \int_0^T \|B^* w_2\|_G^2 dt \leq e_{-1}(W_1)(0) + e_0(W_2)(0), \quad (2.1.56)$$

and

for all $T > T_3$

$$e_{-1}(W_1)(0) \leq C_2 \int_0^T \|B^* w_2\|_G^2 dt, \quad (2.1.57)$$

$$e_0(W_2)(0) \leq C_2 \int_0^T \|B^* w_2\|_G^2 dt, \quad (2.1.58)$$

with the same constants C_1 and C_2 than in (ii) – (iii).

Remark 2.23. This Lemma shows that Z as above defined satisfies a hidden regularity result, namely that for all initial data $W^0 \in X_{-1}$, $\mathbf{B}^* Z_2 \in L^2([0, T]; G)$.

Proof. Since $W \in \mathcal{C}^0([0, T]; H_{-1}^2 \times H_{-2}^2)$, $w_1 \in \mathcal{C}([0, T]; H_{-1})$. Thanks to assumption (A2), $C_{21}^* \in \mathcal{L}(H_1)$, thus we have $C_{21} \in \mathcal{L}(H_{-1})$, thus w_2 is a solution of

$$\begin{cases} w_2'' + Aw_2 = -C_{21}w_1 \in \mathcal{C}([0, T]; H_{-1}), \\ (w_2)_{|t=T} = w_2^T \in H, \\ (w_2')_{|t=T} = q_2^T \in H_{-1}, \end{cases}$$

so that $(w_2, w_2') \in \mathcal{C}([0, T]; H \times H_{-1})$ by uniqueness. This yields $W \in \mathcal{C}^0([0, T]; X_{-1})$.

We set $Z = \mathcal{A}^{-1}W$. Thanks to (2.1.16) together with properties (i) and (iii) in Lemma 2.20, we easily deduce (2.1.53). This proves (ii).

Thanks to (ii) in Lemma 2.20, we have

$$e_0(W_2)(0) \leq C \left(e_0(Z_1)(0) + e_1(Z_2)(0) \right).$$

This together with (2.1.17) yield

$$e_0(W_2)(0) \leq C \left(d_1(T) + d_2(T) \right) \int_0^T \|\mathbf{B}^* Z_2\|_G^2 dt.$$

On the other hand, thanks to (i) in Lemma 2.20 and to (2.1.17), we have

$$e_{-1}(W_1)(0) \leq d_1(T) \int_0^T \|\mathbf{B}^* Z_2\|_G^2 dt.$$

Thus (2.1.54)-(2.1.55) hold with $C_2 = \max \left(C(d_1(T) + d_2(T)), d_1(T) \right)$. This proves (iii).

The properties (iv) follow easily from the hypothesis on $\mathbf{B}^*(w, w')$ and from the definition of Z which implies that $z'_2 = w_2$. \square

Lemma 2.24. *Assume (A1) – (A5). Let $T > 0$ be given. For $W^T = (w_1^T, w_2^T, q_1^T, q_2^T) \in X_1$, we denote by $W = (w_1, w_2, w'_1, w'_2)$ the unique solution in $\mathcal{C}^0([0, T]; H_{-1}^2 \times H_{-2}^2)$ of (2.1.52) Then $W \in \mathcal{C}^0([0, T]; X_1)$.*

Proof. The proof is similar to the proof of Lemma 2.22 and is left to the reader. \square

2.2 Controllability of 2-coupled cascade hyperbolic systems by a single control

We apply the HUM method [34, 30] to deduce from the indirect observability inequality obtained in the previous section, an indirect exact controllability result for the control problem. Prior to this, we will recall for the sake of completeness, the transposition method (see [34, 30]) which allows to define the solutions of the control problem.

2.2.1 The transposition method for the scalar wave equation

We consider the control problem

$$\begin{cases} y'' + Ay = Bv, \\ (y, y')(0) = (y^0, y^1), \end{cases} \quad (2.2.59)$$

where A satisfies (A1). We consider two cases:

- (i) either $B \in \mathcal{L}(G; H)$ (bounded control operator). In this case, we define $\mathbf{B}^*(w, w') = B^*w'$. We also set $\mathcal{F}_{-1} = H \times H_{-1}$ and $\mathcal{F}_{-1}^* = H_1 \times H$.
- or (ii) $B \in \mathcal{L}(G, H'_2)$ (unbounded control operator). In this case, we define $\mathbf{B}^*(w, w') = B^*w$. We also set $\mathcal{F}_1 = H_1 \times H$ and $\mathcal{F}_1^* = H \times H_{-1}$.

We assume that \mathbf{B}^* satisfies the assumption (A4).

Definition 2.25. • *Case (i): Let us first assume that $\mathbf{B}^*(w, w') = B^*w'$ with $B \in \mathcal{L}(G; H)$. Let $(y^0, y^1) \in \mathcal{F}_{-1}^*$ and $v \in L_{loc}^2([0, \infty); G)$ be fixed arbitrarily. We say that (y, y') is a solution by transposition of (2.2.59) if $(y, y') \in \mathcal{C}([0, \infty); \mathcal{F}_{-1}^*)$ satisfies*

$$\langle y'(T), w_T^0 \rangle_{H, H} - \langle y(T), w_T^1 \rangle_{H_1, H_{-1}} = \langle y^1, w(0) \rangle_{H, H} - \langle y^0, w'(0) \rangle_{H_1, H_{-1}} + \int_0^T \langle v, B^*w \rangle_G dt \quad \forall T > 0, \forall (w_T^0, w_T^1) \in \mathcal{F}_{-1}, \quad (2.2.60)$$

where w is the solution of

$$\begin{cases} w'' + Aw = 0, \\ (w, w')(T) = (w_T^0, w_T^1). \end{cases} \quad (2.2.61)$$

• *Case (ii): Let us assume now that $\mathbf{B}^*(w, w') = B^*w$ where $B \in \mathcal{L}(G, H_2')$. Let $(y^0, y^1) \in \mathcal{F}_1^*$ and $v \in L_{loc}^2([0, \infty); G)$ be fixed arbitrarily. We say that (y, y') is a solution by transposition of (2.2.59) if $(y, y') \in \mathcal{C}([0, \infty); \mathcal{F}_1^*)$ satisfies*

$$\langle y'(T), w_T^0 \rangle_{H_{-1}, H_1} - \langle y(T), w_T^1 \rangle_{H, H} = \langle y^1, w(0) \rangle_{H_{-1}, H_1} - \langle y^0, w'(0) \rangle_{H, H} + \int_0^T \langle v, B^*w \rangle_G dt \quad \forall T > 0, \forall (w_T^0, w_T^1) \in \mathcal{F}_1, \quad (2.2.62)$$

where w is the solution of (2.2.61).

Remark 2.26. It is well-known that thanks to (A4), there exists a unique solution by transposition to (2.2.59) which depends continuously on the data (y^0, y^1) and on v . Let us state give a sketch of the proof for the sake of completeness. The solution w of (2.2.61) is such that $(w(0), w'(0))$ depends continuously on $(w_T^0, w_T^1) \in \mathcal{F}_{-1}$ (resp. on $(w_T^0, w_T^1) \in \mathcal{F}_1$) in the case (i) (resp. in the case (ii)). Let us assume that we are in the case (i). Thanks to (A4) with $f = 0$ and applied to $Z = (z, z') = (-A^{-1}w', w)$, we have

$$\int_0^T \|\mathbf{B}^*(z, z')\|_G^2 dt = \int_0^T \|B^*w\|_G^2 dt \leq C_{Te_1}(Z)(0) = C_T \|(w(0), w'(0))\|_{H \times H_{-1}}^2 \leq C_T \|(w_T^0, w_T^1)\|_{H \times H_{-1}}^2,$$

where C_T is a generic constant which depends on T . Hence the right hand side of (2.2.60) defines a continuous linear form with respect to $(w_T^0, w_T^1) \in \mathcal{F}_{-1}$, and moreover this linear form depends continuously on T for all $T > 0$. This implies that for all $T > 0$ there exists a unique solution to (2.2.60) and that this solution depends continuously on T . Let us now assume that we are in the case (ii). Then, thanks to (A4) with $f = 0$ and applied to W , we have

$$\int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt = \int_0^T \|B^*w\|_G^2 dt \leq C_{Te_1}(W)(0) = C_T \|(w(0), w'(0))\|_{H_1 \times H}^2 \leq C_T \|(w_T^0, w_T^1)\|_{H_1 \times H}^2,$$

where C_T is a generic constant which depends on T . Thus, the right hand side of (2.2.62) defines a continuous linear form with respect to $(w_T^0, w_T^1) \in \mathcal{F}_1$, and moreover this linear form depends continuously on T for all $T > 0$. This implies that for all $T > 0$ there exists a unique solution to (2.2.62) and that this solution depends continuously on T .

Remark 2.27. We can, without loss of generality, reverse the time, changing t in $T - t$ in the dual problem (2.2.61) in the above definition, that is we can fix the initial data instead of fixing the final data (with the appropriate minor changes).

Remark 2.28. Note also that we can equivalently replace 0 on the right hand side of (2.2.61) by any f in $L^2((0, T); H_{-1})$ in the case (i) (resp. in $L^2((0, T); H)$ in the case (ii)) for any $T > 0$. More precisely, in the case (i), we can define a solution by transposition by setting

$$\begin{aligned} \langle y'(T), w_T^0 \rangle_{H, H} - \langle y(T), w_T^1 \rangle_{H_1, H_{-1}} + \int_0^T \langle y, f \rangle_{H_1, H_{-1}} dt &= \langle y^1, w(0) \rangle_{H, H} - \\ \langle y^0, w'(0) \rangle_{H_1, H_{-1}} + \int_0^T \langle v, B^* w \rangle_G dt \quad \forall T > 0, \forall (w_T^0, w_T^1) \in \mathcal{F}_{-1}, \forall f \in L^2((0, T); H_{-1}), \end{aligned} \quad (2.2.63)$$

where w is the solution of

$$\begin{cases} w'' + Aw = f, \\ (w, w')(T) = (w_T^0, w_T^1). \end{cases} \quad (2.2.64)$$

In the case (ii), we can define a solution by transposition by setting

$$\begin{aligned} \langle y'(T), w_T^0 \rangle_{H_{-1}, H_1} - \langle y(T), w_T^1 \rangle_{H, H} + \int_0^T \langle y, f \rangle_{H, H} dt &= \langle y^1, w(0) \rangle_{H_{-1}, H_1} - \\ \langle y^0, w'(0) \rangle_{H, H} + \int_0^T \langle v, B^* w \rangle_G dt \quad \forall T > 0, \forall (w_T^0, w_T^1) \in \mathcal{F}_1, \forall f \in L^2((0, T); H), \end{aligned} \quad (2.2.65)$$

where w is the solution of (2.2.64). In both cases, one has to consider respectively the right hand sides of (2.2.63) and (2.2.65) as continuous linear forms in the appropriate spaces, with respect to (w_T^0, w_T^1, f) .

2.2.2 The transposition method for cascade control systems

We consider the control problem

$$\begin{cases} y_1'' + Ay_1 + C_{21}^* y_2 = 0, \\ y_2'' + Ay_2 = Bv, \\ (y_i, y_i')(0) = (y_i^0, y_i^1) \text{ for } i = 1, 2, \end{cases} \quad (2.2.66)$$

where either $B \in \mathcal{L}(G; H)$ (bounded control operator) or $B \in \mathcal{L}(G, H_2')$ (unbounded control operator). We can easily adapt the definition of solutions by transposition to this coupled system in the sense below.

Definition 2.29. • (i) Let $\mathbf{B}^*(w, w') = B^*w'$ with $B \in \mathcal{L}(G, H)$. We set

$$X_{-1}^* = H_2 \times H_1 \times H_1 \times H. \quad (2.2.67)$$

We assume that \mathbf{B}^* satisfies (A4). We say that $(y_1, y_2, y'_1, y'_2) \in \mathcal{C}([0, \infty); X_{-1}^*)$ is a solution by transposition of (2.2.66) if

$$\begin{aligned} \int_0^T \langle v, B^*w_2 \rangle_G dt &= \langle y'_1(T), w_1(T) \rangle_{H_1, H_{-1}} - \langle y_1(T), w'_1(T) \rangle_{H_2, H_{-2}} + \\ &\quad \langle y'_2(T), w_2(T) \rangle_{H, H} - \langle y_2(T), w'_2(T) \rangle_{H_1, H_{-1}} - \\ &\quad \left(\langle y_1^1, w_1(0) \rangle_{H_1, H_{-1}} - \langle y_1^0, w'_1(0) \rangle_{H_2, H_{-2}} + \right. \\ &\quad \left. \langle y_2^1, w_2(0) \rangle_{H, H} - \langle y_2^0, w'_2(0) \rangle_{H_1, H_{-1}} \right), \forall T > 0, \forall W^T \in X_{-1}, \end{aligned} \quad (2.2.68)$$

where $W = (w_1, w_2, w'_1, w'_2)$ is the solution of

$$\begin{cases} w_1'' + Aw_1 = 0, \\ w_2'' + Aw_2 + C_{21}w_1 = 0, \\ W|_{t=T} = W^T, \end{cases} \quad (2.2.69)$$

• (ii) Let $\mathbf{B}^*(w, w') = B^*w$ with $B \in \mathcal{L}(G, H_2')$. We set

$$X_1^* = H_1 \times H \times H \times H_{-1}. \quad (2.2.70)$$

We assume that \mathbf{B}^* satisfies (A4). We say that $(y_1, y_2, y'_1, y'_2) \in \mathcal{C}([0, \infty); X_1^*)$ is a solution by transposition of (2.2.66) if

$$\begin{aligned} \int_0^T \langle v, B^*w_2 \rangle_G dt &= \langle y'_1(T), w_1(T) \rangle_{H, H} - \langle y_1(T), w'_1(T) \rangle_{H_1, H_{-1}} + \\ &\quad \langle y'_2(T), w_2(T) \rangle_{H_{-1}, H_1} - \langle y_2(T), w'_2(T) \rangle_{H, H} - \\ &\quad \left(\langle y_1^1, w_1(0) \rangle_{H, H} - \langle y_1^0, w'_1(0) \rangle_{H_1, H_{-1}} + \right. \\ &\quad \left. \langle y_2^1, w_2(0) \rangle_{H_{-1}, H_1} - \langle y_2^0, w'_2(0) \rangle_{H, H} \right), \forall T > 0, \forall W^T \in X_1, \end{aligned} \quad (2.2.71)$$

where $W = (w_1, w_2, w'_1, w'_2)$ is the solution of (2.2.69).

Remark 2.30. The dual problem (2.2.69) is not conservative as for the case of the dual problem for a single wave equation. However the solution of the non homogeneous equation in w_2 depends continuously on the final data for w_2 and on the source term w_1 , which itself depends continuously on the final data for w_1 . This implies the existence of a unique solution in the sense of transposition of the above control problem in cascade, which has the desired regularity and depends continuously on the initial data and on the control v .

Remark 2.31. Assume that (y_1, y_2, y'_1, y'_2) is a solution by transposition of (2.2.66) (in the required space depending on case (i) or (ii)). We can choose final data W^T such that the final data for w_1 are vanishing in (2.2.69). Hence $w_1 \equiv 0$ and w_2 is any solution of the homogeneous scalar wave equation (in the appropriate space). We deduce then easily that (y_2, y'_2) is a solution by transposition of the scalar equation

$$\begin{cases} y_2'' + Ay_2 = Bv, \\ (y_2, y'_2)(0) = (y_2^0, y_2^1). \end{cases} \quad (2.2.72)$$

Using the Remark 2.28 with $f = -C_{21}w_1$ for the definition of the solution by transposition y_2 for (2.2.72), together with the definition 2.29 for the solutions by transposition for the cascade control system, we deduce the following properties

- (i) If $\mathbf{B}^*(w, w') = B^*w'$ with $B \in \mathcal{L}(G, H)$, then (y_1, y'_1) satisfies

$$\begin{aligned} \langle y'_1(T), w_1(T) \rangle_{H_1, H_{-1}} - \langle y_1(T), w'_1(T) \rangle_{H_2, H_{-2}} - \\ \left(\langle y_1^1, w_1(0) \rangle_{H_1, H_{-1}} - \langle y_1^0, w'_1(0) \rangle_{H_2, H_{-2}} \right) = \\ - \int_0^T \langle y_2, C_{21}w_1 \rangle_{H_1, H_{-1}}, \forall T > 0, \forall W_1^T \in H_{-1} \times H_{-2}, \end{aligned} \quad (2.2.73)$$

where $W_1 = (w_1, w'_1)$ is the solution of

$$\begin{cases} w_1'' + Aw_1 = 0, \\ (w_1, w'_1)|_{t=T} = W_1^T = (w_{1,T}^0, w_{1,T}^1). \end{cases} \quad (2.2.74)$$

- (ii) If $\mathbf{B}^*(w, w') = B^*w$ with $B \in \mathcal{L}(G, H'_2)$, then (y_1, y'_1) satisfies

$$\begin{aligned} \langle y'_1(T), w_1(T) \rangle_{H, H} - \langle y_1(T), w'_1(T) \rangle_{H_1, H_{-1}} - \\ \left(\langle y_1^1, w_1(0) \rangle_{H, H} - \langle y_1^0, w'_1(0) \rangle_{H_1, H_{-1}} \right) = \\ - \int_0^T \langle y_2, C_{21}w_1 \rangle_{H, H}, \forall T > 0, \forall W_1^T \in H \times H_{-1}, \end{aligned} \quad (2.2.75)$$

where $W_1 = (w_1, w'_1)$ is the solution of (2.2.74).

The converse is also true. If y_2 solves (2.2.72) and y_1 satisfies (2.2.73) (resp. (2.2.75)) in the case (i) (resp. (ii)), then (y_1, y_2, y'_1, y'_2) is the corresponding solution by transposition of the cascade system (2.2.66).

2.2.3 Controllability results for cascade systems

Theorem 2.32. Assume the hypotheses (A1)–(A5). We define $T_3 > 0$ as in Theorem 2.7. We have the following properties.

- (i) Let $\mathbf{B}^*(w, w') = B^*w'$ where $B \in \mathcal{L}(G, H)$ is such that (A4) – (A5) holds. Then, for all $T > T_3$, and all $Y_0 \in X_{-1}^*$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (2.2.66) satisfies $Y(T) = 0$.
- (ii) Let $\mathbf{B}^*(w, w') = B^*w$ where $B \in \mathcal{L}(G, H'_2)$ is such that (A4) – (A5) holds. Then, for all $T > T_3$, and all $Y_0 \in X_1^*$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (2.2.66) satisfies $Y(T) = 0$.

Proof. We first consider the case (i). Let $Y^0 = (y_1^0, y_2^0, y_1^1, y_2^1) \in X_{-1}^*$. We consider the bilinear form Λ on X_{-1} defined by

$$\Lambda(W^T, \widetilde{W}^T) = \int_0^T \langle B^*w_2, B^*\widetilde{w}_2 \rangle_G dt, \forall W^T, \widetilde{W}^T \in X_{-1}, \quad (2.2.76)$$

and the linear form on X_{-1} defined for all $W^T \in X_{-1}$ by

$$\mathcal{L}(W^T) = \langle y_1^1, w_1(0) \rangle_{H_1, H_{-1}} - \langle y_1^0, w'_1(0) \rangle_{H_2, H_{-2}} + \langle y_2^1, w_2(0) \rangle_{H, H} - \langle y_2^0, w'_2(0) \rangle_{H_1, H_{-1}}, \quad (2.2.77)$$

where $W = (w_1, w_2, w'_1, w'_2)$ and $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}'_1, \widetilde{w}'_2)$ are respectively solutions of (2.1.52) and (2.2.69). Thanks respectively to the admissibility inequality (2.1.56) and to the observability inequalities (2.1.57)-(2.1.58) with T replacing 0, Λ is continuous and coercive on X_{-1} for $T > T_3$. From the usual energy estimates for the time reverse problem for $Z = \mathcal{A}^{-1}W$, and the conservation of $e_0(Z_1)$ through time we have

$$e_0(Z_1)(0) + e_1(Z_2)(0) \leq C(e_0(Z_1)(T) + e_1(Z_2)(T)). \quad (2.2.78)$$

This together with Lemma 2.20-(iv), lead to

$$e_{-1}(W_1)(0) + e_0(W_2)(0) \leq C(e_{-1}(W_1)(T) + e_0(W_2)(T)),$$

so that \mathcal{L} is continuous on X_{-1} . Hence, thanks to Lax-Milgram Lemma, there exists a unique $W^T \in X_{-1}$ such that

$$\Lambda(W^T, \widetilde{W}^T) = -\mathcal{L}(\widetilde{W}^T), \quad \forall \widetilde{W}^T \in X_{-1}. \quad (2.2.79)$$

We set $v = B^*w_2$. Then, thanks to the hidden regularity property due to (2.1.56), we have $v \in L^2([0, T]; G)$. Thus, we have by definition of the solution of (2.2.66) by transposition

$$\begin{aligned} \int_0^T \langle v, B^*\widetilde{w}_2 \rangle_G dt &= \langle y'_1(T), \widetilde{w}_1(T) \rangle_{H_1, H_{-1}} - \langle y_1(T), \widetilde{w}'_1(T) \rangle_{H_2, H_{-2}} + \langle y'_2(T), \widetilde{w}_2(T) \rangle_{H, H} - \\ &\quad \langle y_2(T), \widetilde{w}'_2(T) \rangle_{H_1, H_{-1}} - \mathcal{L}(\widetilde{W}^T), \forall \widetilde{W}^T \in X_{-1}. \end{aligned} \quad (2.2.80)$$

On the other hand, we have

$$\int_0^T \langle v, B^*\widetilde{w}_2 \rangle_G dt = \Lambda(W^T, \widetilde{W}^T) = -\mathcal{L}(\widetilde{W}^T),$$

so that, we deduce from these two relations that $Y(T) = (y_1, y_2, y'_1, y'_2)(T) = 0$.

Assume now that (ii) holds. Let $Y_0 \in X_1^*$. We consider on X_1 the bilinear form

$$\Lambda(U^T, \tilde{U}^T) = \int_0^T \langle B^* u_2, B^* \tilde{u}_2 \rangle_G dt, \forall U^T, \tilde{U}^T \in X_1, \quad (2.2.81)$$

and the linear form on X_1 defined by

$$\mathcal{L}(U^T) = \langle y_1^1, u_1(0) \rangle_{H,H} - \langle y_1^0, u'_1(0) \rangle_{H_1, H_{-1}} + \langle y_2^1, u_2(0) \rangle_{H_{-1}, H_1} - \langle y_2^0, u'_2(0) \rangle_{H,H}, \forall U^T \in X_1. \quad (2.2.82)$$

Thanks respectively to the admissibility inequality (2.1.16) and to the observability inequality (2.1.17), Λ is continuous and coercive on X_1 for $T > T_3$. On the other hand \mathcal{L} is continuous on X_1 thanks to (2.2.78) with U replacing Z . Hence, thanks to Lax-Milgram Lemma, there exists a unique $U^T \in X_1$ such that

$$\Lambda(U^T, \tilde{U}^T) = -\mathcal{L}(\tilde{U}^T), \quad \forall \tilde{U}^T \in X_1. \quad (2.2.83)$$

We set $v = B^* u_2$. We deduce as for the case (i) that $Y(T) = 0$. \square

2.2.4 Further generalizations

We can change the functional setting in Theorem 2.32 to smoother or weaker control spaces in which null controllability holds, under the appropriate hypotheses replacing the assumptions (A2) – (A5). Let $k \in \mathbb{Z}$ be given. We consider the following new set of assumptions.

$$(A2)_k \begin{cases} C_{21}^* \in \mathcal{L}(H_p) \text{ for } p \in \{-k, 1-k\}, \\ \|A^{k/2} C_{21} A^{-k/2}\| = \beta, |A^{k/2} C_{21} A^{-k/2} w|^2 \leq \beta \langle A^{k/2} C_{21} A^{-k/2} w, w \rangle \quad \forall w \in H, \\ \exists \alpha > 0 \text{ such that, } \alpha |\Pi w|^2 \leq \langle A^{k/2} C_{21} A^{-k/2} w, w \rangle \quad \forall w \in H, \end{cases} \quad (2.2.84)$$

where the operator Π satisfies the assumptions (A3).

We make the following assumptions on the observability operator \mathbf{B}^* .

$$(A4)_k \begin{cases} \mathbf{B}^* \in \mathcal{L}(H_{k+2} \times H_k; G), \\ \forall T > 0 \exists C > 0, \text{ such that for all } (w^0, w^1) \in H_{k+1} \times H_k \text{ and all } f \in L^2([0, T]; H_k), \\ \text{the solution } w \text{ of } w'' + Aw = f, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \leq C \left(e_{k+1}(W)(0) + e_{k+1}(W)(T) + \int_0^T e_{k+1}(W)(t) dt + \right. \\ \left. \int_0^T |A^{k/2} f|^2 dt \right), \end{cases} \quad (2.2.85)$$

where $W = (w, w')$.

We further assume the following observability inequality for a single equation

$$(A5)_k \begin{cases} \exists T_0 > 0, \forall T > T_0, \exists C_1(T) > 0 \text{ such that} \\ \forall (w^0, w^1) \in H_{k+1} \times H_k, \text{ the solution } w \text{ of} \\ w'' + Aw = 0, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \geq C_1(T) e_{k+1}(W)(0). \end{cases} \quad (2.2.86)$$

Then we can prove the following result.

Theorem 2.33. *Let $k \in \mathbb{Z}$ be given. We define the sets*

$$X_{-1,k}^* = H_{2-k} \times H_{1-k} \times H_{1-k} \times H_{-k}, \quad X_{1,k}^* = H_{1-k} \times H_{-k} \times H_{-k} \times H_{-k-1}.$$

Assume the hypotheses (A1), (A2)_k, (A3) and (A4)_k – (A5)_k. We define $T_3 > 0$ as in Theorem 2.7. Then, we have the following properties.

- (i) *Let $\mathbf{B}^*(w, w') = B^*w'$ where $B \in \mathcal{L}(G, H'_k)$ is such that (A4)_k – (A5)_k holds. Then, for all $T > T_3$, and all $Y_0 \in X_{-1,k}^*$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (2.2.66) satisfies $Y(T) = 0$.*
- (ii) *Let $\mathbf{B}^*(w, w') = B^*w$ where $B \in \mathcal{L}(G, H'_{k+2})$ is such that (A4)_k – (A5)_k holds. Then, for all $T > T_3$, and all $Y_0 \in X_{1,k}^*$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (2.2.66) satisfies $Y(T) = 0$.*

We only sketch the proof since it follows that of the previous section on admissibility and observability for the dual cascade system, and that of Theorem 2.32 with the appropriate changes of unknowns, of the observability operator and of the coupling term.

Proof. We define an operator $\overline{\mathbf{B}}^* \in \mathcal{L}(H_2 \times H; G)$ as follows.

$$\overline{\mathbf{B}}^*(\overline{u}, \overline{v}) = \mathbf{B}^*(A^{-k/2}\overline{u}, A^{-k/2}\overline{v}) \quad \forall (\overline{u}, \overline{v}) \in H_2 \times H.$$

We consider the dual problem

$$\begin{cases} u_1'' + Au_1 = 0, \\ u_2'' + Au_2 + C_{21}u_1 = 0, \\ (u_i, u'_i)(0) = (u_i^0, u_i^1) \in H_{k+2} \times H_k \text{ for } i = 1, 2. \end{cases} \quad (2.2.87)$$

We introduce the new unknowns $\overline{u}_i = A^{k/2}u_i$ for $i = 1, 2$. We set $\overline{C}_{21} = A^{k/2}C_{21}A^{-k/2}$. Then we have $\overline{\mathbf{B}}^*\overline{U}_2 = \mathbf{B}^*U_2$, where $\overline{U}_2 = (\overline{u}_2, \overline{u}'_2)$. Moreover $\overline{U} = (\overline{u}_1, \overline{u}_2, \overline{u}'_1, \overline{u}'_2)$ solves (2.1.1) in \mathcal{H} , where \overline{C}_{21} replaces C_{21} . By construction and thanks to (A4)_k – (A5)_k, the observability operator $\overline{\mathbf{B}}^*$ satisfies (A4) – (A5) with f replaced by $\overline{f} = A^{k/2}f$ and $W = (w, w')$ replaced by $\overline{W} = (A^{k/2}w, A^{k/2}w')$ in (A4). Moreover \overline{C}_{21} and Π satisfy respectively (A2) and (A3). Hence we can apply Lemma 2.5 and Theorem 2.7 to \overline{U} with the observability operator $\overline{\mathbf{B}}^*$ and the coupling \overline{C}_{21} .

Coming back to the unknown U and to the observability operator \mathbf{B}^* , we deduce the following admissibility property

$$\int_0^T \|\mathbf{B}^* U_2\|_G^2 dt \leq C \left(e_k(U_1)(0) + e_{k+1}(U_2)(0) \right), \quad (2.2.88)$$

for all initial data $U^0 \in H_{k+1}^2 \times H_k^2$, and the following observability inequalities for all $T > T_3$, and all initial data $U^0 \in H_{k+1}^2 \times H_k^2$,

$$\begin{cases} d_1(T) \int_0^T \|\mathbf{B}^* U_2\|_G^2 \geq e_k(U_1)(0), \\ d_2(T) \int_0^T \|\mathbf{B}^* U_2\|_G^2 \geq e_{k+1}(U_2)(0), \end{cases} \quad (2.2.89)$$

where T_3 , $d_i(T)$ for $i = 1, 2$ are as in Theorem 2.7.

This allows us to show that the HUM operator is well-defined in the case (i) on the set

$$X_{-1,k} = H_{k-1} \times H_k \times H_{k-2} \times H_{k-1},$$

and in the case (ii) on the set

$$X_{1,k} = H_k \times H_{k+1} \times H_{k-1} \times H_k,$$

By duality, the control problem is well-posed in $X_{-1,k}^*$ (resp. in $X_{1,k}^*$) for the case (i) (resp. (ii)). \square

Remark 2.34. *It is easy to provide examples of operators C_{21} satisfying $(A2)_k$. It is for instance sufficient to choose $C_{21} = A^{-k/2} D A^{k/2}$ where D satisfies $(A2)$.*

Remark 2.35. *A further generalization of Theorem 2.32 is to consider a more general control system of the form (2.2.66), but including additionally source terms ξ_1 and ξ_2 in appropriate spaces (for well-posedness) on respectively the first and second equation of (2.2.66) as considered in subsection 3.3. The above controllability Theorem 2.32 can then be extended with no difficulties to handle these source terms, modifying the HUM method as in the proof of Theorem 3.11.*

3 Main applicative results

We shall give in this section the main consequences on the most well-known examples of applications, namely: wave-type, heat-type and Schrödinger cascade coupled systems.

Let Ω be a bounded open set in \mathbb{R}^d with a sufficiently smooth boundary Γ . The set Ω can also be a smooth connected compact Riemannian manifold with or without boundary as in [4]. Let T be a given positive time. We recall the following definition for the Geometric Control Condition of Bardos Lebeau Rauch [11].

Definition 3.1. *We say that an open subset ω of Ω satisfies (GCC) if there exists a time $T > 0$ such that every generalized geodesic traveling at speed 1 in Ω meets ω at a time $t < T$. We say that a subset Γ_1 of the boundary Γ satisfies (GCC) if there exists a time $T > 0$ such that every generalized geodesic traveling at speed 1 in Ω meets Γ_1 at a time $t < T$ in a non-diffractive point.*

3.1 Boundary and localized observability/controllability of 2-coupled cascade wave equations with localized couplings

3.1.1 Main observability results for 2-coupled cascade wave systems

We consider the following 2-coupled cascade system of wave equations

$$\begin{cases} u_{1,tt} - \Delta u_1 = 0 & \text{in } (0, T) \times \Omega, \\ u_{2,tt} - \Delta u_2 + c_{21}(x)u_1 = 0 & \text{in } (0, T) \times \Omega, \\ u_i = 0 \text{ for } i = 1, 2 & \text{in } (0, T) \times \partial\Omega, \\ (u_i, u_{i,t})(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2 & \text{in } \Omega, \end{cases} \quad (3.1.90)$$

where the subscript t denotes the partial derivative with respect to time t . We set $H = L^2(\Omega)$, and we consider the operator A defined by $Au = -\Delta u$ for $u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. We set $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1)$ for all the sequel of this section. We make the following assumptions on the coefficient c_{21} .

$$(H1) \begin{cases} c_{21} \in W^{1,\infty}(\Omega) \\ c_{21} \geq 0 \text{ on } \Omega, \\ \{c_{21} > 0\} \supset \overline{O_2} \text{ for some open subsets } O_2 \subset \Omega. \end{cases}$$

Remark 3.2. *The coupling term c_{21} is assumed to be strictly positive on $\overline{O_2}$, so that the coupling effect due to this term is effective in a neighborhood of this set. We will say in all the sequel that the subset $\overline{O_2}$ is the region on which the coupling is localized.*

We recall that

$$D(A^{k/2}) = \{u \in H^k(\Omega), u = \Delta u = \dots \Delta^{[(k-1)/2]} u = 0 \text{ on } \Gamma\} \quad (3.1.91)$$

for all $k \in \{0, 1, \dots\}$, where $[x]$ stands for the integer part of the real number x .

We do not recall below the admissibility property for the coupled system which allows to show that the observations of the solution are well-defined (hidden regularity result) in a classical way. This property is given in the abstract Theorem 2.7.

Theorem 3.3 (Observability estimates). *We assume that the hypothesis (H1) holds for some open subset $O_2 \subset \Omega$ that satisfies (GCC). Then we have the following results.*

- (i) *Locally distributed observation. Let b_2 be a given function defined on Ω such that*

$$(H2) \begin{cases} b_2 \in \mathcal{C}(\overline{\Omega}), b_2 \geq 0 \text{ on } \Omega, \\ \{b_2 > 0\} \supset \overline{\omega_2} \text{ for some open subset } \omega_2 \subset \Omega, \end{cases}$$

where ω_2 satisfies (GCC). Then there exists $T^ > 0$ such that for all $T > T^*$, there exist constants $c_{i,2}(T) > 0$, $i = 1, 2$ such that for all $U^0 \in L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$, the following observability inequalities hold*

$$c_{1,2}(T) \|(u_1^0, u_1^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq \int_0^T \int_{\Omega} b_2 |u_{2,t}|^2 dx dt,$$

$$c_{2,2}(T) \|(u_2^0, u_2^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq \int_0^T \int_{\Omega} b_2 |u_{2,t}|^2 dx dt.$$

- (ii) *Boundary observability.* Let b_2 be a given function defined on Γ such that

$$(H3) \begin{cases} b_2 \in C(\overline{\Gamma}), b_2 \geq 0 \text{ on } \Gamma \text{ for } , \\ \{b_2 > 0\} \supset \overline{\Gamma_2} \text{ for some subset } \Gamma_2 \subset \Gamma, \end{cases}$$

where Γ_2 satisfies (GCC). Then there exists $T^* > 0$ such that for all $T > T^*$, there exist constants $c_{i,2}(T) > 0$, $i = 1, 2$ such that for all $U^0 \in L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ the following observability inequalities hold

$$c_{1,2}(T) \|(u_1^0, u_1^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq \int_0^T \int_{\Gamma} b_2 \left| \frac{\partial u_2}{\partial \nu} \right|^2 d\sigma dt,$$

$$c_{2,2}(T) \|(u_2^0, u_2^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq \int_0^T \int_{\Gamma} b_2 \left| \frac{\partial u_2}{\partial \nu} \right|^2 d\sigma dt.$$

Remark 3.4. As in the Remark 3.2, the subsets ω_2 (resp. Γ_2) of ω (resp. Γ) are the regions (indeed in a neighborhood of them) on which the observations are localized. Hence, we prove that 2-coupled cascade systems with a coupling term localized on a subregion $O_2 \subset \Omega$ with a single observation either locally distributed on $\omega_2 \subset \Omega$ or distributed on a part $\Gamma_2 \subset \Gamma$ of the boundary, is observable under the geometric condition that both O_2 and ω_2 (resp. O_2 and Γ_2) satisfy (GCC). Hence this covers many situations for which the intersection $\overline{O_2} \cap \overline{\omega_2} = \emptyset$ (resp. $\overline{O_2} \cap \overline{\Gamma_2} = \emptyset$). As shown in Theorem 2.8 –which gives a necessary and sufficient condition for the observability of 2-coupled system– this condition is sharp, and even necessary when $\partial\Omega$ has no contact of infinite order with its tangent, or is analytic. Moreover, we can exhibit many situations for which this sharp condition holds with $\overline{O_2} \cap \overline{\omega_2} = \emptyset$ (resp. $\overline{O_2} \cap \overline{\Gamma_2} = \emptyset$).

3.1.2 Main controllability results for 2-coupled cascade wave systems

We now consider the 2-coupled control cascade system subjected to a single either locally distributed or a boundary control. We shall start with the case of a locally distributed control.

$$\begin{cases} y_{1,tt} - \Delta y_1 + c_{21} y_2 = 0 \text{ in } (0, T) \times \Omega, \\ y_{2,tt} - \Delta y_2 = b_2 v_2 \text{ in } (0, T) \times \Omega \\ y_i = 0 \text{ in } (0, T) \times \Gamma \text{ for } i = 1, 2, \\ (y_i, y_{i,t})(0) = (y_i^0, y_i^1) \text{ for } i = 1, 2. \end{cases} \quad (3.1.92)$$

We set $Y_0 = (y_1^0, y_2^0, y_1^1, y_2^1)$. Then we have the following exact controllability result.

Theorem 3.5. *We assume that the coefficient c_{21} satisfy the hypothesis (H1) for some open subsets $O_2 \subset \Omega$ that satisfies (GCC). We also assume that the coefficients b_2 and the subset ω_2 satisfies (H2) where the subsets ω_2 satisfies (GCC). Then for all $T > T^*$, and all $Y_0 \in \Pi_{i=1}^2 D(A^{(3-i)/2}) \times \Pi_{i=1}^2 D(A^{(2-i)/2})$, there exists a control function $v_2 \in L^2((0, T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (3.1.92) with initial data Y_0 satisfies $Y(T) = 0$.*

We now consider the following 2-coupled control cascade system with a boundary control.

$$\begin{cases} y_{1,tt} - \Delta y_1 + c_{21}y_2 = 0 & \text{in } (0, T) \times \Omega, \\ y_{2,tt} - \Delta y_2 = 0 & \text{in } (0, T) \times \Omega \\ y_1 = 0 & \text{in } (0, T) \times \Gamma, y_2 = b_2 v_2 & \text{in } (0, T) \times \Gamma, \\ (y_i, y_{i,t})(0) = (y_i^0, y_i^1) & \text{for } i = 1, \dots, 2. \end{cases} \quad (3.1.93)$$

Then we have the following exact controllability result.

Theorem 3.6. *We assume that the coefficient c_{21} satisfies the hypothesis (H1) for some open subset $O_2 \subset \Omega$ that satisfies (GCC). We also assume that the coefficient b_2 and the subset Γ_2 satisfy (H3) where the subset Γ_2 satisfies (GCC). Then for all $T > T^*$, and all $Y_0 \in \Pi_{i=1}^2 D(A^{(2-i)/2}) \times \Pi_{i=1}^2 D(A^{(1-i)/2})$, there exists a control function $v_2 \in L^2((0, T); L^2(\Gamma))$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (3.1.93) with initial data Y_0 satisfies $Y(T) = 0$.*

3.2 Boundary and localized controllability of 2-coupled cascade heat and Schrödinger equations with localized couplings

We now consider the following 2-coupled locally control cascade heat-type system. We shall first consider the case of a locally distributed control.

$$\begin{cases} e^{i\theta} y_{1,t} - \Delta y_1 + c_{21}y_2 = 0 & \text{in } (0, T) \times \Omega, \\ e^{i\theta} y_{2,t} - \Delta y_2 = b_2 v_2 & \text{in } (0, T) \times \Omega \\ y_i = 0 & \text{in } (0, T) \times \Gamma \text{ for } i = 1, 2, \\ y_i(0) = y_i^0 & \text{in } \Omega \text{ for } i = 1, 2. \end{cases} \quad (3.2.94)$$

We set $Y_0 = (y_1^0, y_2^0)$. We recover 2-coupled heat (resp. Schrödinger) cascade systems when $\theta = 0$ (resp. $\theta = \pm\pi/2$) and diffusive coupled cascade systems when $\theta \in (-\pi/2, \pi/2)$. Then we have the following exact controllability result.

Corollary 3.7. *We assume that the coefficient c_{21} satisfies the hypothesis (H1) for some open subset $O_2 \subset \Omega$ that satisfies (GCC). We also assume that the coefficient b_2 and the subset ω_2 satisfy (H2) where the subset ω_2 satisfies (GCC). Then, the following properties hold*

- (i) *The case $\theta \in (-\pi/2, \pi/2)$ (Heat type systems). We have for all $T > 0$, and all $Y_0 \in (L^2(\Omega))^2$, there exist a control function $v_2 \in L^2((0, T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2)$ of (3.2.94) with initial data Y_0 satisfies $Y(T) = 0$.*

- (ii) The case $\theta = \pm\pi/2$ (Schrödinger systems). We have for all $T > 0$ and all $Y_0 \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a control function $v_2 \in L^2((0, T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2)$ of (3.2.94) with initial data Y_0 satisfies $Y(T) = 0$.

We now consider the following 2-coupled control cascade heat-type system with a boundary control.

$$\begin{cases} e^{i\theta} y_{1,t} - \Delta y_1 + c_{21} y_2 = 0 & \text{in } (0, T) \times \Omega, \\ e^{i\theta} y_{2,t} - \Delta y_2 = 0 & \text{in } (0, T) \times \Omega \\ y_1 = 0 & \text{in } (0, T) \times \Gamma, y_2 = b_2 v_2 & \text{in } (0, T) \times \Gamma, \\ y_i(0) = y_i^0 & \text{in } \Omega \text{ for } i = 1, 2. \end{cases} \quad (3.2.95)$$

Then we have the following exact controllability result.

Corollary 3.8. *We assume that the coefficients c_{21} satisfies the hypothesis (H1) for some open subset $O_2 \subset \Omega$ that satisfies (GCC). We also assume that the coefficients b_2 and the subsets Γ_2 satisfy (H3) where the subset Γ_2 satisfies (GCC). Then we have*

- (i) The case $\theta \in (-\pi/2, \pi/2)$ (Heat type systems). We have for all $T > 0$, and all $Y_0 \in (H^{-1}(\Omega))^2$, there exist a control function $v_2 \in L^2((0, T); L^2(\Gamma))$ such that the solution $Y = (y_1, y_2)$ of (3.2.95) with initial data Y_0 satisfies $Y(T) = 0$.
- (ii) The case $\theta = \pm\pi/2$ (Schrödinger systems). We have for all $T > 0$ and all $Y_0 \in L^2(\Omega) \times H^{-1}(\Omega)$, there exist a control function $v_2 \in L^2((0, T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2)$ of (3.2.95) with initial data Y_0 satisfies $Y(T) = 0$.

Remark 3.9. *It should be noted that (GCC) is not a natural condition for the scalar heat equation for which null-controllability holds for arbitrary nonempty control region. This property probably still holds for coupled cascade system. In fact, when $O_2 \cap \omega_2 \neq \emptyset$, condition GCC is not needed (see [24]). Hence the results presented here are not optimal as far as geometric conditions are concerned. They are optimal for one-dimensional domains. Since Jaffard's results [28] on internal controllability/observability of Schrödinger equation in a rectangle, it is also known that (GCC) is not a necessary condition for Schrödinger equation (see also [46]). However the results presented above are valid in a multidimensional setting, for boundary as well as locally distributed control and for geometric situations for which the control region does not meet the coupling region.*

Remark 3.10. *The above observability and controllability results also hold true if the operator $-\Delta$ is replaced by a more general uniformly elliptic operator, or by higher order operators such as the bilaplacian for instance.*

3.3 Insensitizing controls for the wave equation

In this part we give new results on the existence of insensitizing controls for the wave equation, which can be either locally distributed controls or boundary controls. This problem reads as follows. We consider the scalar wave equation with a locally distributed control v (the localization of the control depends on the the support of

the coefficient function b):

$$\begin{cases} y_{tt} - \Delta y = \xi + bv & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{in } (0, T) \times \Gamma, \\ y(0, \cdot) = y^0 + \tau_0 z^0 \text{ in } \Omega, y_t(0, \cdot) = y^1 + \tau_1 z^1 \text{ in } \Omega, \end{cases} \quad (3.3.96)$$

and the scalar wave equation with a boundary control v (the localization of the control depends on the support of the coefficient function b in Γ):

$$\begin{cases} y_{tt} - \Delta y = \xi & \text{in } (0, T) \times \Omega, \\ y = bv & \text{in } (0, T) \times \Gamma, \\ y(0, \cdot) = y^0 + \tau_0 z^0 \text{ in } \Omega, y_t(0, \cdot) = y^1 + \tau_1 z^1 \text{ in } \Omega, \end{cases} \quad (3.3.97)$$

where for both cases, the inhomogeneity $\xi \in L^2((0, T) \times \Omega)$, the initial data (y^0, y^1) are given known functions in $H_0^1(\Omega) \times L^2(\Omega)$ or in $L^2(\Omega) \times H^{-1}(\Omega)$, whereas the perturbations z^0, z^1 are supposed to be unknown of norm 1 in the appropriate spaces. Here the real numbers τ_0, τ_1 are assumed to be small and to measure the amplitudes of the unknown perturbations of the initial data. We associate to the solutions y of (3.3.96) (resp. to (3.3.97)) the functional defined by

$$\Phi(y, \tau_0, \tau_1) = \frac{1}{2} \int_0^T \int_{\Omega} c y^2 dx dt, \quad (3.3.98)$$

where c is a function which has a localized support in the neighborhood of a subset O which is a given subset of Ω . The functional Φ consists in a “weighted” observation of the solution on the set O during a length of time T . It is an observation when the coefficient $c = \mathbb{1}_O$. The control bv is said to insensitize the “weighted” observation Φ if for all (z^0, z^1) the corresponding solution y of (3.3.96) (resp. (3.3.97)) satisfies

$$\frac{\partial \Phi}{\partial \tau_0}(y, 0, 0) = \frac{\partial \Phi}{\partial \tau_1}(y, 0, 0) = 0. \quad (3.3.99)$$

We shall see that (3.3.99) is equivalent to solve an appropriate exact controllability result for a cascade system, namely for (4.3.103) (resp. (4.3.106)) for the locally distributed (resp. boundary) insensitizing control problem. We refer to J.-L. Lions [35] for the introduction of this notion and its reduction to a controllability problem of an associated cascade system. This notion was further weakened to a notion of ε -insensitizing control for parabolic equations by Bodart and Fabre [12]. The first results concerning the existence of boundary insensitizing (resp. ε - localized insensitizing) controls for the wave equation have been obtained by Dáger [19] in the case of the one-dimensional wave equation. They have been extended by Tebou [44] to the multidimensional wave equation in the case of locally distributed control, and for coupling regions which meet the control region. The case of localized insensitizing control for a semilinear wave equation with a different observation (namely the trace of the normal derivative of the solution) has been considered in [45]. Positive results have been obtained in [45] under the strong geometric condition that the boundary observation region satisfies the usual multiplier geometric condition and that the localized control region contains a neighbourhood of the observation region.

Here we are interested in general results for (3.3.96) and (3.3.97) in sharp geometric situations, in particular we want positive results in situations for which $\text{supp}\{c\} \cap \text{supp}\{b\} = \emptyset$.

Theorem 3.11. Assume that c satisfies

$$\begin{cases} c \in W^{1,\infty}(\Omega), c \geq 0 \text{ on } \Omega, \\ \{c > 0\} \supset \overline{O} \text{ for some open subset } O \subset \Omega. \end{cases} \quad (3.3.100)$$

We have the following properties

- *Locally distributed control.* Let b be given such that

$$\begin{cases} b \in C(\overline{\Omega}), b \geq 0 \text{ on } \Omega, \\ \{b > 0\} \supset \overline{\omega} \text{ for some open subset } \omega \subset \Omega. \end{cases} \quad (3.3.101)$$

We assume that O and ω satisfy (GCC). Then for any given $\xi \in L^2((0, T); L^2(\Omega))$ and $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists an exact control $v \in L^2((0, T); L^2(\Omega))$ that drives back the solution of the scalar wave equation

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi + bv & \text{in } (0, T) \times \Omega, \\ y_2 = 0 & \text{in } (0, T) \times \Gamma, \\ (y_2, y_{2,t})|_{t=0} = (y^0, y^1) & \text{in } \Omega, \end{cases}$$

to equilibrium, i.e. $y_2(T) = y_{2,t}(T) = 0$ and insensitizes Φ along the solutions of (3.3.96), i.e. v is such that (3.3.99) holds for any $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

- *Boundary control.* Let b be given such that

$$\begin{cases} b \in C(\overline{\Gamma}), b \geq 0 \text{ on } \Gamma, \\ \{b > 0\} \supset \overline{\Gamma_1} \text{ for some subset } \Gamma_1 \subset \Gamma. \end{cases} \quad (3.3.102)$$

We assume that O and Γ_1 satisfy (GCC). Then for any given $\xi \in L^2((0, T); L^2(\Omega))$ and $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists an exact control $v \in L^2((0, T); L^2(\Gamma))$ that drives back the solution of the scalar wave equation

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi & \text{in } (0, T) \times \Omega, \\ y_2 = bv & \text{in } (0, T) \times \Gamma, \\ (y_2, y_{2,t})|_{t=0} = (y^0, y^1) & \text{in } \Omega, \end{cases}$$

to equilibrium, i.e. $y_2(T) = y_{2,t}(T) = 0$ and insensitizes Φ along the solutions of (3.3.97), i.e. v is such that (3.3.99) holds for any $(z^0, z^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

- Assume that $\partial\Omega$ has no contact of infinite order with its tangent, or is analytic. Then the above condition on O and ω (resp. on O and Γ_1) for the case of locally (resp. boundary) distributed control are necessary.

4 Proofs of the main applicative results

4.1 Proofs of the results for coupled cascade wave systems

We begin with the proof of Theorem 3.3.

Proof. of Theorem 3.3. This is an application of the abstract Theorem 2.7. Here $H = L^2(\Omega)$, and A is given by $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The sets $H_k = D(A^{k/2})$ are given by (3.1.91). The operator C_{21} is the multiplication operator in $L^2(\Omega)$ by the corresponding function c_{21} and is therefore bounded and self-adjoint. Thanks to the smoothness assumptions on the coefficients c_{21} , $C_{21} \in \mathcal{L}(H_k)$ for all $k = 0, 1$. Thanks to (H1), the assumption (A2) holds with $\Pi = \mathbb{1}_{O_2}$ and $\alpha = \inf_{O_2}(c_{21}) > 0$. Moreover since the set O_2 satisfies (GCC), (A3) also holds. We shall check the assumptions on the observability operator. First case: Locally distributed observation. Then we have $G = G_2 = L^2(\Omega)$, and $B^* = B_2^*$ is the multiplication in $L^2(\Omega)$ by the bounded function b_2 . Therefore B_2^* is a bounded symmetric operator in H , so that (A4) holds with $B^* = B_2^*$. Thanks to the assumptions on b_2 in the case of locally distributed observation and since ω_2 satisfies (GCC), we deduce that (A5) also holds using [11].

Second case: Boundary observability. Then we have $G = G_2 = L^2(\Gamma)$, and $B^* = B_2^* \in \mathcal{L}(H^2(\Omega) \cap H_0^1(\Omega); L^2(\Gamma))$ is defined as

$$B_2^* u = b_2 \frac{\partial u}{\partial \nu} \quad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Thanks to the well-known hidden regularity result of [34], $B^* = B_2^*$ satisfies (A4). Thanks to the assumptions on b_2 in the case of boundary observability and since Γ_2 satisfies (GCC), we deduce that (A5) also holds using [11]. Hence we can apply Theorem 2.7, which gives the desired result. \square

We now shall prove the controllability results for coupled cascade wave systems.

Proof. of Theorem 3.5. Thanks to our hypotheses and thanks to the above proof the assumption of Theorem 2.7 are satisfied. Hence we apply the part (i) of Theorem 2.32. This gives the desired result. \square

Proof. of Theorem 3.6. Thanks to our hypotheses and thanks to the above proof, the assumption of Theorem 2.7 are satisfied. Hence we apply the part (ii) of Theorem 2.32. This gives the desired result. \square

4.2 Proofs of the results for coupled cascade heat and Schrödinger equations

We start with the proof of Corollary 3.7 using the transmutation method.

Proof. of Corollary 3.7. We proceed as in [4]. Proof of (i). We first apply no control on the time interval $(0, T/2)$, so that by the smoothing effect of the heat equation, the initial data $Y^0 \in (L^2(\Omega))^2$ is driven to $Y_{t=T/2} \in \Pi_{i=1}^2 D(A^{(3-i)/2})$. We then combine Theorem 3.5 together with the transmutation result given by Miller in [37] to prove that there exists a control v_2 such that $Y(T) = 0$.

Proof of (ii). It is similar to the case (i) except that we work directly on the time interval $(0, T)$ since no smoothing effect holds in the case of the Schrödinger equation. Combining Theorem 3.5 together with the transmutation result given by Miller in [38] (see Theorem 3.1), we conclude the proof. \square

Proof. of Corollary 3.8. We proceed as in [4]. Proof of (i). We first apply no control on the time interval $(0, T/2)$, so that by the smoothing effect of the heat equation, the initial data $Y^0 \in (H^{-1}(\Omega))^2$ is driven to $Y_{|t=T/2} \in \Pi_{i=1}^2 D(A^{(2-i)/2})$. We then combine Theorem 3.6 together with the transmutation result given by Miller in [39] (see Theorem 3.4) to get the desired result.

Proof of (ii). It is similar to the case (i) except that we work directly on the time interval $(0, T)$ since no smoothing effect holds in the case of the Schrödinger equation. Combining Theorem 3.6 together with the transmutation result (Theorem 3.1) given by Miller in [38] (see Theorem 3.1), we conclude the proof. \square

4.3 Proofs of the results on insensitizing controls for the wave equation

The proof of Theorem 3.11 relies on the appropriate admissibility and observability inequalities for 2-coupled cascade systems given in the Lemmata 2.5, 2.22 and in Theorem 2.7. Namely it relies on the following classical proposition (established in a different form in [19] for the one-dimensional case) linking the existence of insensitizing controls to that of certain properties for an associated cascade system.

Proposition 4.1. *Assume $c \in W^{1,\infty}(\Omega)$, and let $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\xi \in L^2((0, T); L^2(\Omega))$ be given. We have the following properties*

- *Locally distributed control. Let b be given in $\mathcal{C}(\overline{\Omega})$, and locally supported in $\overline{\Omega}$. If $v \in L^2((0, T); L^2(\Omega))$ is such that the solution of the v -controlled cascade system*

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x)y_2 = 0 & \text{in } (0, T) \times \Omega, \\ y_{2,tt} - \Delta y_2 = \xi + bv & \text{in } (0, T) \times \Omega, \\ y_1 = y_2 = 0 & \text{in } (0, T) \times \Gamma, \\ Y^0 = (y_1, y_2, y_{1,t}, y_{2,t})|_{t=0} = (0, y^0, 0, y^1) & \text{in } \Omega, \end{cases} \quad (4.3.103)$$

satisfies

$$y_1(T) = y_{1,t}(T) = 0 \text{ in } \Omega, \quad (4.3.104)$$

then v insensitizes Φ along the solutions of (3.3.96). In particular if the control v can be chosen such that the controlled cascade system (4.3.103) satisfies $Y(T) = (y_1(T), y_2(T), y_{1,t}(T), y_{2,t}(T)) = (0, 0, 0, 0)$, then v is an exact control for the scalar wave equation

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi + bv & \text{in } (0, T) \times \Omega, \\ y_2 = 0 & \text{in } (0, T) \times \Gamma, \\ (y_2, y_{2,t})|_{t=0} = (y^0, y^1) & \text{in } \Omega, \end{cases} \quad (4.3.105)$$

which both drives the solution of the scalar wave equation (4.3.105) to equilibrium and insensitizes Φ along the solutions of (3.3.96).

Conversely, if v insensitizes Φ along the solutions of (3.3.96), then the solution of (4.3.103) satisfies (4.3.104).

- *Boundary control.* Let b be given in $\mathcal{C}(\overline{\Gamma})$, and locally supported in Γ . If $v \in L^2((0, T); L^2(\Gamma))$ is such that the solution of the v -controlled cascade system

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x)y_2 = 0 & \text{in } (0, T) \times \Omega, \\ y_{2,tt} - \Delta y_2 = \xi & \text{in } (0, T) \times \Omega, \\ y_1 = 0, y_2 = bv & \text{in } (0, T) \times \Gamma, \\ Y^0 = (y_1, y_2, y_{1,t}, y_{2,t})|_{t=0} = (0, y^0, 0, y^1) & \text{in } \Omega, \end{cases} \quad (4.3.106)$$

satisfies (4.3.104),

then, v insensitizes Φ along the solutions of (3.3.97). In particular if the control v can be chosen such that the controlled cascade system (4.3.106) satisfies $Y(T) = (y_1(T), y_2(T), y_{1,t}(T), y_{2,t}(T)) = (0, 0, 0, 0)$, then v is an exact control for the scalar wave equation

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi & \text{in } (0, T) \times \Omega, \\ y_2 = bv & \text{in } (0, T) \times \Gamma, \\ (y_2, y_{2,t})|_{t=0} = (y^0, y^1) & \text{in } \Omega, \end{cases} \quad (4.3.107)$$

which both drives the solution of the scalar wave equation (4.3.107) to equilibrium and insensitizes Φ along the solutions of (3.3.97).

Conversely if v insensitizes Φ along the solutions of (3.3.97), then the solution of (4.3.106) satisfies (4.3.104).

Proof. We first consider the case of locally distributed control. Thanks to our assumptions on b, ξ, y^0, y^1 the equation in y_2 has a unique solution $y_2 \in \mathcal{C}([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$. Thus in a similar way the equation in y_1 has a unique solution $y_1 \in \mathcal{C}([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; H_0^1(\Omega))$. Assume that the solution of (4.3.103) satisfies (4.3.104). We have

$$\frac{\partial \Phi}{\partial \tau_0}(y, 0, 0) = \int_0^T \int_{\Omega} c y_2 \hat{w} \, dx \, dt, \quad (4.3.108)$$

and

$$\frac{\partial \Phi}{\partial \tau_1}(y, 0, 0) = \int_0^T \int_{\Omega} c y_2 \hat{z} \, dx \, dt, \quad (4.3.109)$$

for all $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ (of norm 1 in these spaces), where \hat{w}, \hat{z} are the respective solutions of

$$\begin{cases} \hat{w}_{tt} - \Delta \hat{w} = 0 & \text{in } (0, T) \times \Omega, \\ \hat{w} = 0 & \text{in } (0, T) \times \Gamma, \\ \hat{w}(0, \cdot) = z^0 \text{ in } \Omega, \hat{w}_t(0, \cdot) = 0 \text{ in } \Omega, \end{cases} \quad (4.3.110)$$

$$\begin{cases} \hat{z}_{tt} - \Delta \hat{z} = 0 & \text{in } (0, T) \times \Omega, \\ \hat{z} = 0 & \text{in } (0, T) \times \Gamma, \\ \hat{z}(0, \cdot) = 0 \text{ in } \Omega, \hat{z}_t(0, \cdot) = z^1 \text{ in } \Omega. \end{cases} \quad (4.3.111)$$

We set

$$w(t, \cdot) = \hat{w}(T - t, \cdot), z(t, \cdot) = \hat{z}(T - t, \cdot) \quad \forall t \in [0, T].$$

Then w and z solves respectively

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } (0, T) \times \Omega, \\ w = 0 & \text{in } (0, T) \times \Gamma, \\ w(T, \cdot) = z^0 \text{ in } \Omega, w_t(T, \cdot) = 0 \text{ in } \Omega, \end{cases} \quad (4.3.112)$$

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } (0, T) \times \Omega, \\ z = 0 & \text{in } (0, T) \times \Gamma, \\ z(T, \cdot) = 0 \text{ in } \Omega, z_t(T, \cdot) = -z^1 \text{ in } \Omega. \end{cases} \quad (4.3.113)$$

We set $\hat{y}_i(t) = y_i(T - t)$ for $t \in [0, T]$ and $i = 1, 2$. Then, by definition of the solution by transposition (see (2.2.73) in Remark 2.31) and thanks to the relations $\hat{y}_1(T) = \hat{y}_{1,t}(T) = 0$, we have

$$-\int_0^T \langle \hat{y}_2, cw \rangle_{H_1, H_{-1}} dt = -\langle \hat{y}_{1,t}(0), w(0) \rangle_{H_1, H_{-1}} + \langle \hat{y}_1(0), w_t(0) \rangle_{H_2, H_{-2}}.$$

Since $\hat{y}_1(0) = \hat{y}_{1,t}(0) = 0$ and $w(t) \in L^2(\Omega)$ for all $t \in [0, T]$, we deduce that

$$\int_0^T \int_{\Omega} c \hat{y}_2 w dx dt = \int_0^T \int_{\Omega} c y_2 \hat{w} dx dt = 0 \quad \forall z^0 \in H_0^1(\Omega).$$

In a similar way, replacing w by z in the above relation, we deduce that

$$\int_0^T \int_{\Omega} c y_2 \hat{z} dx dt = 0 \quad \forall z^1 \in L^2(\Omega).$$

Hence v insensitizes Φ along the solutions of (3.3.96).

Conversely assume that v insensitizes Φ along the solutions of (3.3.96). We denote by y_2 the solution of

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi + bv & \text{in } (0, T) \times \Omega, \\ y_2 = 0 & \text{in } (0, T) \times \Gamma, \\ (y_2, y_{2,t})|_{t=0} = (y^0, y^1) & \text{in } \Omega. \end{cases}$$

Then

$$\int_0^T \int_{\Omega} c y_2 \hat{w} dx dt = \int_0^T \int_{\Omega} c y_2 \hat{z} dx dt = 0,$$

for all $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ (of norm 1 in these spaces), where \hat{w}, \hat{z} are the respective solutions of (4.3.110) and (4.3.111). We define as before $w(t) = \hat{w}(T - t)$, $z(t) = \hat{z}(T - t)$, $\hat{y}_2(t) = y_2(T - t)$ for $t \in [0, T]$. Moreover we introduce the solution \hat{y}_1 of

$$\begin{cases} \hat{y}_{1,tt} - \Delta \hat{y}_1 + c \hat{y}_2 = 0 & \text{in } (0, T) \times \Omega, \\ \hat{y}_1 = 0 & \text{in } (0, T) \times \Gamma, \\ (\hat{y}_1, \hat{y}_{1,t})|_{t=0} = (0, 0) & \text{in } \Omega. \end{cases}$$

Then, $(\hat{y}_1, \hat{y}_2, \hat{y}'_1, \hat{y}'_2)$ is the solution of a cascade system of the form (4.3.103) with control $v(T - \cdot)$, source term $\xi(T - \cdot)$ and initial data $(\hat{y}_1, \hat{y}_2, \hat{y}_{1,t}, \hat{y}_{2,t})|_{t=0} = (0, y_2(T), 0, -y_{2,t}(T))$. By definition of the solutions by transposition of this cascade system (see (2.2.73)), we have

$$0 = - \int_0^T \int_{\Omega} c \hat{y}_2 w \, dx \, dt = - \int_0^T \langle \hat{y}_2, cw \rangle_{H_1, H_{-1}} \, dt = \langle \hat{y}_{1,t}(T), w(T) \rangle_{H_1, H_{-1}} - \langle \hat{y}_1(T), w_t(T) \rangle_{H_2, H_{-2}} - \left(\langle \hat{y}_{1,t}(0), w(0) \rangle_{H_1, H_{-1}} - \langle \hat{y}_1(0), w_t(0) \rangle_{H_2, H_{-2}} \right).$$

Since the initial data for \hat{y}^1 are vanishing and since $w(T) = z^0$ and $w_t(T) = 0$, we have

$$0 = - \int_0^T \langle \hat{y}_2, cw \rangle_{H_1, H_{-1}} \, dt = \langle \hat{y}_{1,t}(T), z^0 \rangle_{H_1, H_{-1}} \quad \forall z^0 \in H_0^1(\Omega).$$

Hence by reflexivity of H_1 , and density of $H_1 = H_0^1(\Omega)$ in H_{-1} , we deduce that $\hat{y}_{1,t}(T) = 0$. In a similar way replacing w by z and since $z(T) = 0$ and $z_t(T) = -z^1$, we have

$$0 = - \int_0^T \langle \hat{y}_2, cz \rangle_{H_1, H_{-1}} \, dt = \langle \hat{y}_1(T), z^1 \rangle_{H_2, H_{-2}} \quad \forall z^1 \in L^2(\Omega).$$

Hence by reflexivity of H_2 and density of $H = L^2(\Omega)$ in H_{-2} , we deduce that $\hat{y}_1(T) = 0$. We set $y_1(t) = \hat{y}_1(T - t)$ for $t \in [0, T]$. Then (y_1, y_2, y'_1, y'_2) is the solution of (4.3.103) and satisfies (4.3.104).

The proof of the above results is similar in the boundary control case using the definition of transposition solutions and is left to the reader. \square

Proof. of Theorem 3.11. We first consider the case of distributed control. Let $Y^0 = (0, y^0, 0, y^1) \in X_{-1}^*$. We consider the bilinear form Λ on X_{-1} defined by

$$\Lambda(W^T, \widetilde{W}^T) = \int_0^T \int_{\Omega} b w_2 b \widetilde{w}_2 \, dx \, dt.$$

and the continuous linear form on X_{-1} defined for all $W^T \in X_{-1}$ by

$$\mathcal{L}(W^T) = \langle y^1, w_2(0) \rangle_{H, H} - \langle y^0, w_{2,t}(0) \rangle_{H_1, H_{-1}}, \quad (4.3.114)$$

where $W = (w_1, w_2, w_{1,t}, w_{2,t})$ is the solution of

$$\begin{cases} w_{1,tt} - \Delta w_1 = 0, \\ w_{2,tt} - \Delta w_2 + c w_1 = 0, \\ W|_{t=T} = W^T, \end{cases} \quad (4.3.115)$$

and $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_{1,t}, \widetilde{w}_{2,t})$ is the solution of (4.3.115) where W^T is replaced by \widetilde{W}^T . We also define a linear form on X_{-1} by

$$J(W^T) = \int_0^T \int_{\Omega} \xi w_2 \, dx \, dt. \quad (4.3.116)$$

Thanks to the inequalities (2.1.19) and (2.1.56) applied to Z_2 where $Z = \mathcal{A}_2^{-1}W$, we deduce that J is continuous on X_{-1} . Thanks respectively to the admissibility inequality (2.1.56) and to the observability inequalities (2.1.57)-(2.1.58), Λ is continuous and coercive on X_{-1} for $T > T_3$. Hence, thanks to Lax-Milgram Lemma, there exists a unique $W^T \in X_{-1}$ such that

$$\Lambda(W^T, \widetilde{W}^T) = -\mathcal{L}(\widetilde{W}^T) - J(\widetilde{W}^T), \quad \forall \widetilde{W}^T \in X_{-1}. \quad (4.3.117)$$

We set $v = bw_2$. Then $v \in L^2([0, T]; L^2(\Omega))$ and we have by definition of the solution of (4.3.103) by transposition

$$\begin{aligned} \int_0^T \int_{\Omega} v b \widetilde{w}_2 dx dt &= \langle y_{1,t}(T), \widetilde{w}_1(T) \rangle_{H_1, H_{-1}} - \langle y_1(T), \widetilde{w}_{1,t}(T) \rangle_{H_2, H_{-2}} + \langle y_{2,t}(T), \widetilde{w}_2(T) \rangle_{H, H} - \\ &\quad \langle y_2(T), \widetilde{w}_{2,t}(T) \rangle_{H_1, H_{-1}} - \mathcal{L}(\widetilde{W}^T) - J(\widetilde{W}^T), \quad \forall \widetilde{W}^T \in X_{-1}. \end{aligned}$$

On the other hand, we have

$$\int_0^T \int_{\Omega} v b \widetilde{w}_2 dx dt = \Lambda(W^T, \widetilde{W}^T) = -\mathcal{L}(\widetilde{W}^T) - J(\widetilde{W}^T),$$

so that, we deduce from these two relations that $Y(T) = (y_1, y_2, y'_1, y'_2)(T) = 0$. In particular we have $y_1(T) = y_{1,t}(T) = 0$ where (y_1, y_2) is the solution of (4.3.103). Applying Proposition 4.1, we conclude that v insensitizes Φ along the solutions of (3.3.96). Since $y_2(T) = y_{2,t}(T) = 0$ holds, v is also an exact control for the scalar wave equation (4.3.105) solved by y_2 .

We now turn to the case of boundary control. Let $Y_0 = (0, y^0, 0, y^1) \in X_1^*$. Thanks to the hidden regularity property induced by the admissibility inequality (2.1.53), the bilinear form

$$\Lambda(U^T, \widetilde{U}^T) = \int_0^T \int_{\Gamma} b \frac{\partial u_2}{\partial \nu} b \frac{\partial \widetilde{u}_2}{\partial \nu} d\sigma dt, \quad \forall U^T, \widetilde{U}^T \in X_1, \quad (4.3.118)$$

is well-defined and continuous on X_1 . We consider the linear form on X_1 defined by

$$\mathcal{L}(U^T) = \langle y^1, u_2(0) \rangle_{H_{-1}, H_1} - \langle y^0, u_{2,t}(0) \rangle_{H, H}, \quad \forall U^T \in X_1. \quad (4.3.119)$$

Thanks respectively to the admissibility inequality (2.1.16) and to the observability inequality (2.1.17), Λ is continuous and coercive on X_1 for $T > T_3$. Thanks to (2.1.19) and to (2.1.16), J is continuous on X_1 . Hence, thanks to Lax-Milgram Lemma, there exists a unique $U^T \in X_1$ such that

$$\Lambda(U^T, \widetilde{U}^T) = -\mathcal{L}(\widetilde{U}^T) - J(\widetilde{U}^T), \quad \forall \widetilde{U}^T \in X_1. \quad (4.3.120)$$

We set $v_2 = b \frac{\partial u_2}{\partial \nu}$. We deduce as for the previous case that $Y(T) = 0$. Hence $y_1(T) = y_{1,t}(T) = 0$ where (y_1, y_2) is the solution of (4.3.106). Applying Proposition 4.1, we conclude that v insensitizes Φ along the solutions of (3.3.97). Since $y_2(T) = y_{2,t}(T) = 0$ holds, v is also an exact control for the scalar wave equation (4.3.107) solved by y_2 . \square

5 Conclusion and perspectives

This paper presents several results that solve partially or completely, open questions on the control of 2-coupled cascade wave, heat or Schrödinger systems. In particular, we give a necessary and sufficient condition on the coupling and control coefficients for the observation of the full cascade hyperbolic system by a single, either bounded or unbounded observation. This leads to results for the existence of insensitizing controls for the scalar wave equation. Such controls are built in a such a way that a given localized measure of the solution, is robust to small unknown variations of the initial data. We give a complete answer, under the form of a necessary and sufficient condition for the existence of such insensitizing controls. In the case of spatial domains such that their boundaries have no contact of infinite order with its tangent (or are analytic), this necessary and sufficient condition says that the (either internal or boundary) control region and the localized measure region should satisfy both the Geometric Control Condition [11].

Our results are based on an extension of the two-level energy method introduced in [1, 2] for 2-symmetric coupled systems (see also [3, 4] in the case of partially coercive coupling operators), to 2-coupled cascade systems. Indeed 2-coupled symmetric systems are conservative, whereas the total energy of 2-coupled cascade system is not preserved through time.

The method presented in this paper does not provide the minimal control time for the full coupled system, on the other hand it is a constructive method. In [33], the authors characterize the minimal control time in the case of locally distributed observation in a compact manifold without boundary, by contradiction arguments. It would be interesting to have characterizations of the minimal control time in the case of boundary observation, and to give its expression in simple geometric conditions. It would also be interesting to characterize this minimal control time by direct and constructive arguments.

Many other questions are challenging. For instance, it is important to extend the present results to n -coupled cascade systems of coupled PDE's. We have several recent results in this direction but for length reasons and for the sake of clarity, we present these further results on n -coupled cascade systems in a different paper [7]. These results have for instance applications on the simultaneous control of devices coupled in parallel.

Further generalizations are to determine whether if it would be possible to extend our positive results to cascade systems with different operators A_1 and A_2 for the coupled system (2.1.1). The first results in this direction are given for symmetric coupled systems in [2] in situations for which $A_1 = -r_1\Delta$, $A_2 = -r_2\Delta$ with $r_1 \neq r_2$ (with restrictions on r_1/r_2) and under geometric restrictions and for specific forms of couplings. We also provide positive examples with operators of different order such as $A_1 = -\Delta$ and $A_2 = \Delta^2$ allowing to couple different PDE's such as wave and plate equations. Negative results for 2-coupled cascade systems have been obtained in this direction in [33] in the torus for wave and heat one-dimensional cascade systems for two different diffusion coefficients (see also [20]). Such questions have also been considered for 2-coupled parabolic cascade systems in [23] with constant couplings and in the case of boundary control. In this paper, the authors exhibit a one-dimensional parabolic cascade system with two different operators of the form $A_1 = -d_1\Delta$ and $-d_2\Delta$, with $d_1 \neq d_2$ and $\sqrt{\frac{d_1}{d_2}} \in \mathbb{Q}$, for which null controllability does not hold. They also provide examples for which approximate controllability holds whereas null controllability does not hold.

One can also consider the same principal operator A in both equations of (2.1.1), perturbed in each equation

by bounded terms of the form $C_{11}u_1$ and $C_{22}u_2$ respectively in the first and second equations, that is

$$\begin{cases} u_1'' + Au_1 + C_{11}u_1 = 0, \\ u_2'' + Au_2 + C_{22}u_2 + C_{21}u_1 = 0, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2, \end{cases}$$

where C_{11} and C_{22} are bounded operators in the pivot space H . In this case the two operators A_1 and A_2 differ only by bounded perturbations, it would be also interesting to determine necessary and sufficient conditions for observability by a single observation on the second component u_2 . In particular our results can be generalized to the case $C_{11} = C_{11}^* = C_{22}$ with $A + C_{11}$ satisfying the assumption (A1) (it is then sufficient to replace A by $A + C_{11}$ in the proof). Extensions to more general coupling terms is an open question.

Several applications, in particular for reaction-diffusion systems, involve nonlinear control systems. In the spirit of [18], it would be of great interest to extend the present results to nonlinear coupled systems.

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